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THE ELEMENTS

OF

ALGEBRA.

DESIGNED FOR THE USE OF STUDENTS IN THE UNIVERSITY.

BY

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ADVERTISEMENT.

In the perusal of the treatise comprised in the following pages, the only thing required of the student is a competent knowledge of the principles and practice of *Arithmetic:* and upon these, as a groundwork, he will find himself conducted by gradual and easy steps to all the more elementary results of Algebra, considered in the light of Universal Arithmetic, as well as to the more general formulæ of Symbolical Algebra, regarded as the basis of Analytical Investigations. The precise point at which Arithmetical Algebra terminates and Symbolical Algebra commences is not very easily described by any simple definition; but in the course of the work many opportunities have been taken to direct the student's attention to the distinction between them: and he will scarcely fail to recognize the essential differences in the characters of the two, by noticing the interpretations of various results which Arithmetic could not contemplate. Many alterations and additions have been made in the present edition, and the entire substance contained in it is briefly enumerated in the table of contents. The first Appendix comprises a variety of Theorems and Problems with their Solutions, some of which were found in the text of the preceding editions; and the second Appendix is a collection of Examples for Practice, to all of which the Answers or Results have been annexed.

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THE

ELEMENTS OF ALGEBRA.

CHAPTER I.

DEFINITIONS AND PRELIMINARY OBSERVATIONS.

ARTICLE I. DEFINITION.

ALGEBRA, in its simplest Character, is a Method of representing numerical magnitudes by means of *Symbols*, and of expressing their mutual dependance upon each other by means of *Signs*.

On these grounds, it exhibits the relations of quantities in *General Forms*: and where Arithmetic *effects* the solution of any particular Problem, Algebra *indicates*, by an universal language, the solutions of all Arithmetical Questions under the same or similar circumstances.

Viewed in this light only, and as long as its operations and results are strictly accordant with those of Arithmetic, it is termed *Arithmetical Algebra*; but in all other cases it is called *Symbolical Algebra*, and it then gives rise to what is properly styled *Algebraical Analysis*.

This distinction will be duly noticed in the course of the present Treatise.

2. DEF. The *Symbols* employed in this Science are generally the small letters of the Alphabet, a, b, c, &c.; but sometimes the capitals A, B, C, &c., and the letters of the *Greek* Alphabet, α, β, γ, &c., are adopted for the same purposes.

When several quantities are *similarly* employed in any Arithmetical Operation, it is not unusual to represent them by

the *same* letter with the successive natural numbers *suffixed*, as a_0, a_1, a_2, &c.; or by the *same* letter with successive numbers of accents, placed contiguous to it, as a', a'', a''', &c.

3. Def. In the *Investigations* and *Demonstrations* of *Theorems*, the letters of the Alphabet are used indiscriminately: but in the *Solutions* of *Problems*, the *known quantities*, or *data*, are usually denoted by the former letters a, b, c, &c.; and the *unknown quantities*, or *quæsita*, by the latter x, y, z, &c.: this distinction, however, is not always attended to, particularly in the works of the old writers.

4. Def. The *Signs* here made use of are certain *Marks* or *Characters* invented to denote the operations of *Addition*, *Subtraction*, *Multiplication*, *Division*, *Involution* and *Evolution*, which, by reason of the *general forms* of the quantities under consideration, can only sometimes be *effected*, but may always be *indicated* or *expressed*.

5. Def. The sign of *Addition* read *plus* is $+$, and signifies that the quantity, to which it is prefixed, is supposed to be combined with the quantity which precedes it, by the operation of addition.

Thus, in the *Algebraical Expression* $a+b$, the sign $+$ indicates that the quantity represented by b is to be added to that represented by a; but if numerical values were assigned to these symbols, this sign would be no longer necessary to *indicate* an operation which could then be *effected*; for, if a and b represented 7 and 5 respectively, $a+b$ would be equivalent to the *Arithmetical Expression* $7+5$ or 12, in which last the sign has been made to disappear.

The same observations may be made respecting the expression $a+b+c+d$, in which four quantities are understood to be combined in a similar manner; and it is evident that its value will be the same in whatever order the symbols occur: and similarly of more.

Again, $a+a+a$ indicates that *three equal* quantities are to be added together, and the result of this operation, we know, would be *three times* any one of them, or *three times* a.

6. Def. The sign of *Subtraction* called *minus* is $-$, and denotes that the quantity, which it precedes, is understood to be *subtracted* in all cases where the sign $+$ would indicate the operation of *Addition*.

Thus, the expression $a - b$ indicates that the quantity represented by b is to be subtracted from that represented by a: and were the symbols a and b numerically expressed as before, the value of $a - b$ would be $7 - 5$ or 2.

Similar remarks may be applied to such an expression as $a - b + c - d$, in which, by the last article, a and c are understood to be connected by the operation of addition, and from their sum b and d are supposed to be subtracted in succession.

If we take any two expressions, each made up of two or more *Terms*, as $a + b$ and $c + d + e$, and for the sake of keeping them distinct from each other, inclose each in a *Parenthesis* or *Bracket*, or connect their parts by a line called a *Vinculum*, it follows that $(a + b) - (c + d + e)$, or $\overline{a + b} - \overline{c + d + e}$, implies that the latter of these sets of quantities is to be subtracted from the former: and it is of no consequence in what order the letters are placed, provided the expressions retain their proper values.

If $c + d + e$ be greater than $a + b$, the expression $(a + b) - (c + d + e)$ is purely *symbolical*, and can have no place in *Arithmetic*, or *Arithmetical Algebra*.

7. Def. All expressions formed by the operations indicated by the signs $+$ and $-$, are called *Compound* quantities, and the parts of which they are made up are termed *Simple* quantities.

In *Symbolical Algebra*, the signs $+$ and $-$ are regarded as *Qualities* or *Affections* of the symbols to which they are prefixed; and all quantities preceded by the sign $+$, as well as those which have no sign at all, are styled *Additive*, *Positive*, or *Affirmative*; whereas every quantity affected with the sign $-$, is termed *Subtractive* or *Negative*.

It is this circumstance from which originates a characteristic distinction between these two signs, and the signs of the other operations which will be subsequently explained.

8. DEF. The sign of *Multiplication* read *into* is $\times$, and shews that the quantity which precedes it, is intended to be multiplied by that which comes after it.

Thus, $a \times b$ indicates the product of the quantities a and b; and if a and b were numerically expressed by 24 and 6, the multiplication might be effected, and the product would be 24×6 or 144.

Similarly, $a \times b \times c$ denotes the *continued* product of the quantities represented by the letters a, b, c, each of which is called a *Factor:* and the product is said to be of the *Order* or *Degree* expressed by the number of *Letters* it contains, without regard to the *Arithmetical* symbols found in it.

Again, $(a + b - c) \times (h - k + l) \times (x - y - z)$ represents the continued product of the compound factors which it comprises, and it is manifestly immaterial in what order they occur.

In *Algebraical* as well as in *Arithmetical* expressions, this sign is frequently supplied by a *Point:* thus, $a . b$ is equivalent to $a \times b$, $2 . 3$ to 2×3; and more generally in the multiplication of simple *Algebraical* quantities, both signs are entirely omitted, as ab is supposed to be equivalent to either $a \times b$ or $a . b$: so likewise $3 \times a$ and $5 \times b \times x$ may be written $3a$ and $5bx$.

In a product expressed according to these principles, any symbol or symbols, whether Arithmetical or Algebraical, will form what is termed the *Coefficient* of the symbol or symbols which follow it; thus, in $3a$, 3 is the coefficient of a; in $5bx$, $5b$ is the coefficient of x; and in $6abxy$, $6ab$ is the coefficient of xy.

Hence also, the coefficient of a must be understood to be 1.

9. DEF. The sign of *Division* read *by* is $\div$ or $:$, which placed between two quantities, denotes that the former of them is intended to be divided by the latter.

Thus, $a \div b$ or $a : b$, indicates that the quantity represented by a is to be divided by that represented by b: so that, assigning to a and b the numerical values used in the last article, we shall have $a \div b$ equivalent to $24 \div 6$ or 4.

In like manner $(a+b-c) \div (x-y+z)$ denotes the result arising from the division of the former of these compound quantities by the latter.

These signs are but little used, the same operation being more generally expressed by placing the dividend over the divisor with a line between them, according to the form of *Arithmetical Fractions*: thus $\frac{a}{b}$ and $\frac{a+b-c}{x-y+z}$ are equivalent to the expressions just considered.

10. Having now explained the methods of expressing the results of the four *fundamental* operations of Arithmetic as applied to Algebraical Symbols, we shall next endeavour to trace the consequences of the same operations upon algebraical quantities in the forms of Fractions, guided only by the *suggestions* which the established Rules of Arithmetic afford.

Here, referring to the Articles of the *second* edition of the *Author's* Arithmetic as numbered below, we have,

(1) From (77), $\frac{a}{b} = \frac{ac}{bc}$, and $\frac{bd}{cd} = \frac{b}{c}$; by which it appears that the same symbol may be introduced into, or removed from, the numerator and denominator of a fraction, without altering its value.

(2) From (78), $\frac{a}{b} \times c = \frac{ac}{b}$, and $\frac{a}{b} \div c = \frac{a}{bc}$; which indicate the results of the multiplication and division of a fraction, by any new symbol.

(3) From (79), $a + \frac{b}{c} = \frac{ac+b}{c}$, and $a - \frac{b}{c} = \frac{ac-b}{c}$; which are the results, when a mixed quantity is reduced to a fractional form.

(4) From (82), $\frac{ac+b}{c} = a + \frac{b}{c}$, and $\frac{ac-b}{c} = a - \frac{b}{c}$;

which are the converse of the last operations, and express the transformation of a fraction to the form of a mixed quantity.

(5) From (85), $\frac{a}{b} = \frac{ad}{bd}$, and $\frac{c}{d} = \frac{bc}{bd}$, are two fractions expressed in forms having a common denominator: and similarly of more.

(6) The Sum of the fractions $\frac{a}{b}$ and $\frac{c}{d}$, will be expressed by $\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad+bc}{bd}$: and similarly of more.

(7) The Difference of the fractions $\frac{a}{b}$ and $\frac{c}{d}$, will be expressed by $\frac{a}{b} - \frac{c}{d} = \frac{ad}{bd} - \frac{bc}{bd} = \frac{ad-bc}{bd}$.

(8) The Product of the fractions $\frac{a}{b}$ and $\frac{c}{d}$, will be expressed by $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$: and similarly of more.

(9) The Quotient of the fractions $\frac{a}{b}$ and $\frac{c}{d}$, will be expressed by $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$.

(10) The Expression for the *compound fraction* $\frac{a}{b}$ of $\frac{c}{d}$, will be $\frac{ac}{bd}$: and that for the *complex fraction* $\dfrac{a+\frac{b}{c}}{x-\frac{y}{z}}$,

$$\text{will be } \left(a+\frac{b}{c}\right) \div \left(x-\frac{y}{z}\right) = \frac{ac+b}{c} \div \frac{xz-y}{z}$$

$$= \frac{ac+b}{c} \times \frac{z}{xz-y} = \frac{(ac+b)z}{(xz-y)c}.$$

11. Def. The Sign of *Involution* is a small numeral called an *Index* or *Exponent*, placed above the line, a little to the right of the quantity to which it belongs, and is used to represent merely the result of the multiplication of two or more equal factors.

Thus, a^2 is regarded as equivalent to $a \times a$, or aa, which is the product arising from the quantity a being multiplied by itself, and is called the *square* of a, or a *squared.*

Similarly, a^3 is equivalent to $a \times a \times a$, or aaa, and is termed the *cube* of a, or a *cubed.*

The same is supposed to hold whatever be the number of operations employed, as a^m denotes aaa &c., in which the number of equal factors is m, and the number of operations is $m - 1$, and it is called the m^{th} power of a.

Hence also the index of a must be understood to be 1, or a to be equivalent to a^1.

After the same manner, the square, cube, and m^{th} power of a compound quantity, as $ax + b$, are represented by

$$(ax + b)^2, \ (ax + b)^3 \text{ and } (ax + b)^m.$$

12. The powers of $\dfrac{1}{a}$, the *Reciprocal* of a, may, by means of (8) of article (10), be represented in a similar form.

For, the square of $\dfrac{1}{a} = \dfrac{1}{a} \times \dfrac{1}{a} = \dfrac{1}{aa} = \dfrac{1}{a^2}$:

the cube of $\dfrac{1}{a} = \dfrac{1}{a} \times \dfrac{1}{a} \times \dfrac{1}{a} = \dfrac{1}{aaa} = \dfrac{1}{a^3}$:

the m^{th} power of $\dfrac{1}{a} = \dfrac{1}{a} \times \dfrac{1}{a} \times \dfrac{1}{a} \times$ &c. to m factors

$$= \frac{1}{aaa \text{ \&c. to } m \text{ factors}} = \frac{1}{a^m}.$$

Also, the powers of a fraction as $\dfrac{a}{b}$, will admit of the like abbreviated notation.

Thus, the square of $\dfrac{a}{b} = \dfrac{a}{b} \times \dfrac{a}{b} = \dfrac{aa}{bb} = \dfrac{a^2}{b^2}$:

$$\text{the cube of } \frac{a}{b} = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} = \frac{aaa}{bbb} = \frac{a^3}{b^3}:$$

$$\text{the } m^{\text{th}} \text{ power of } \frac{a}{b} = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} \times \text{\&c. to } m \text{ factors}$$

$$= \frac{aaa \text{ \&c. to } m \text{ factors}}{bbb \text{ \&c. to } m \text{ factors}} = \frac{a^m}{b^m}:$$

$$\text{that is, } \left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}, \quad \left(\frac{a}{b}\right)^3 = \frac{a^3}{b^3}, \quad \text{and} \quad \left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}.$$

13. The last two articles lead immediately to certain results constituting what is called the *Theory of Integral Indices*, the generalization of which confers the most important advantages upon the science whereof we are treating.

(1) The product of a^2 and a^3 is expressed by

$$aa \times aaa = aaaaa = a^5:$$

$$\text{that is, } a^2 \times a^3 = a^{2+3} = a^5.$$

Again, to multiply a^m by a^n, we observe that

$$a^m = a \times a \times a \times \text{\&c. to } m \text{ factors}:$$

$$a^n = a \times a \times a \times \text{\&c. to } n \text{ factors}:$$

$$\therefore \text{ the product } a^m \times a^n$$

$$= a \times a \times a \times \text{\&c. to } m \text{ factors} \times a \times a \times a \times \text{\&c. to } n \text{ factors}$$

$$= a \times a \times a \times \text{\&c. to } (m + n) \text{ factors}$$

$= a^{m+n}$, by article (11).

Hence, the product of two powers of any symbol is expressed by the same symbol, with an index equal to the *sum* of the indices of the factors.

Similarly, $a^p \times a^q \times a^r = a^{p+q} \times a^r = a^{p+q+r}$: and so of any number of factors.

(2) The quotient of a^5 by $a^2 = aaaaa \div aa = \dfrac{aaaaa}{aa}$

$= aaa = a^3$, by (1) of article (10):

$$\text{that is, } a^5 \div a^2 = a^{5-2} = a^3.$$

Again, to divide a^m by a^n, we have

$$a^m = a \times a \times a \times \text{\&c. to } m \text{ factors}:$$

$$a^n = a \times a \times a \times \text{\&c. to } n \text{ factors}:$$

therefore, the quotient $a^m \div a^n$, or $\dfrac{a^m}{a^n}$

$$= \frac{a \times a \times a \times \text{\&c. to } m \text{ factors}}{a \times a \times a \times \text{\&c. to } n \text{ factors}}$$

$= a \times a \times a \times$ &c. to $(m - n)$ factors, if m be greater than n,

$= a^{m-n}$, by article (11);

or, $= \dfrac{1}{a \times a \times a \times \text{\&c. to } (n-m) \text{ factors}}$, if m be less than n,

$= \dfrac{1}{a^{n-m}}$, by the same article.

Hence, the quotient of two powers of any symbol, is expressed by the same symbol with an index equal to the *difference* of the indices of the dividend and divisor.

(3) In order that the expression of the result of the division of any one power of a quantity by any other power of the same quantity, may be independent of the *relative* values of their indices, or that the *symbolical* representation of this operation may be *general*, it is evident that a^{m-n} and $\dfrac{1}{a^{n-m}}$ must be regarded as *equivalent* expressions, or expressions having the same meaning; and if this be assumed to be universally true, it is manifest that by making $n = 0$, and $m = 0$, in succession, we shall have a^m equivalent to $\dfrac{1}{a^{-m}}$, and a^{-n} equivalent to $\dfrac{1}{a^n}$.

From this it appears that a factor may always be transferred from the numerator to the denominator, and *vice versâ*, by changing the sign of its index.

(4) In this scheme of notation we have immediately, $1 = \dfrac{a^m}{a^m} = a^{m-m} = a^0$: whence it follows that a^0 may always be regarded as a symbolical representation of *Unit* or 1.

(5) The square of $a^3 = a^3 \times a^3 = a^{3+3} = a^6 = a^{3\times 2}$: that is, $(a^3)^2 = a^{3\times 2} = a^6$.

The cube of $a^2 b$

$= a^2b \times a^2b \times a^2b = a^2 \times a^2 \times a^2 \times b \times b \times b = a^6 b^3 = a^{2\times 3} b^{1\times 3}$:

that is, $(a^2b)^3 = a^{2\times 3} b^{1\times 3} = a^6 b^3$.

The n^{th} power of $a^m = a^m \times a^m \times a^m \times$ &c. to n factors $= a^{m+m+m+\text{\&c. to } n \text{ terms}} = a^{mn}$: that is, $(a^m)^n = a^{mn}$.

Hence, any power of an algebraical quantity is expressed by multiplying the index of each factor comprised in it, by the index of the power proposed.

All these conclusions will evidently hold good whatever be the *form* of a the symbol used; as when $\frac{b}{c}$, $\frac{ax}{y}$, &c. are substituted in the place of a; and inasmuch as the *cipher* or 0, is not excluded from the range of Arithmetical Symbols, we may have $0^0 = 1$, in Symbolical Algebra.

(6) Since by (3) of this article, $a^{-m} = \frac{1}{a^m}$, and $a^{-n} = \frac{1}{a^n}$, we shall manifestly have

$$a^{-m} \times a^{-n} = \frac{1}{a^m} \times \frac{1}{a^n} = \frac{1}{a^{m+n}}, \text{ or } = a^{-m-n}:$$

$$\text{and } a^m \times a^{-n} = a^m \times \frac{1}{a^n} = \frac{a^m}{a^n} = a^{m-n}, \text{ or } = \frac{1}{a^{n-m}}.$$

Also,

$$a^{-m} \div a^{-n} = \frac{1}{a^m} \div \frac{1}{a^n} = \frac{1}{a^m} \times \frac{a^n}{1} = \frac{a^n}{a^m} = a^{n-m}, \text{ or } = \frac{1}{a^{m-n}}:$$

$$\text{and } a^m \div a^{-n} = a^m \div \frac{1}{a^n} = a^m \times a^n = a^{m+n}, \text{ or } = \frac{1}{a^{-m-n}}.$$

$$\text{Again, } (a^{-n})^m = \left(\frac{1}{a^n}\right)^m = \frac{1}{a^{mn}} = a^{-mn}:$$

$$\text{and } (a^n)^{-m} = \frac{1}{(a^n)^m} = \frac{1}{a^{mn}} = a^{-mn}.$$

As before, a may in each of these instances be replaced by a fractional form, as $\frac{b}{c}$, $\frac{ax}{y}$, &c.: and it appears that the results of the operations of Multiplication, Division and Involution are expressed in the *same forms*, whether the indices be positive or negative.

14. DEF. The sign of *Evolution*, called the *Radical Sign*, is $\sqrt{\ }$, and denotes that the root expressed by the numeral which accompanies it, is understood to be extracted from the quantity to which it is prefixed.

Thus, $\sqrt[2]{a}$, which is more generally written $\sqrt{a}$, expresses the *second* root, or *square* root of a: that is, $\sqrt{a}$ denotes the quantity which multiplied by itself produces a.

Similarly, $\sqrt[3]{ab}$ and $\sqrt[m]{a-b}$ represent the *cube* root and the m^{th} root of the quantities ab and $a-b$ respectively.

These operations are more frequently expressed in a manner similar to that in which the powers were denoted in the last article, by means of *fractional indices:* thus $a^{\frac{1}{2}}$, $(ab)^{\frac{1}{3}}$ and $(a-b)^{\frac{1}{m}}$ are considered to be equivalent to $\sqrt{a}$, $\sqrt[3]{ab}$ and $\sqrt[m]{a-b}$, respectively.

In the same manner all other fractions may be used as indices; thus, $a^{\frac{3}{2}}$ is assumed to be of equal signification with $\sqrt{a^3}$, or to represent the *square* root of the *cube* of a; $(a+x)^{\frac{2}{3}}$ to denote the *cube* root of the *square* of $a+x$, and $(a^2-bx)^{\frac{m}{n}}$ the n^{th} root of the m^{th} power of a^2-bx.

Hence, any root of an algebraical quantity is expressed by dividing the index of every factor contained in it, by the number denoting the proposed root.

15. From this method of notation, we must conclude immediately that

$$(a^{\frac{1}{2}})^2 = a,\quad \{(ab)^{\frac{1}{3}}\}^3 = ab,\ \text{and}\ \{(a-b)^{\frac{1}{m}}\}^m = a-b.$$

Also, from analogy we make the general *assumption*,

$$a^{\frac{p}{q}} \times a^{\frac{r}{s}} = a^{\frac{p}{q}+\frac{r}{s}} = a^{\frac{ps+qr}{qs}},$$

where p, q, r and s are any symbols whatever: but when these quantities are whole numbers, it will be proved hereafter that the same result is a necessary consequence of the notation adopted in the last article.

Similarly, when the symbols are general, it is *assumed* that

$$a^{\frac{p}{q}} \div a^{\frac{r}{s}} = a^{\frac{p}{q}-\frac{r}{s}} = a^{\frac{ps-qr}{qs}}, \text{ and } (a^{\frac{p}{q}})^{\frac{r}{s}} = a^{\frac{pr}{qs}}:$$

though, when p, q, r and s are whole numbers, these forms will hereafter be shewn to be capable of arithmetical demonstration.

Again, the m^{th} root of the n^{th} root of $\sqrt[p]{a}$ is denoted by $a^{\frac{1}{mnp}}$; that is, $\sqrt[m]{\sqrt[n]{\sqrt[p]{a}}}$ is equivalent to $a^{\frac{1}{mnp}}$, and so on.

16. DEF. The *Degrees*, *Orders* or *Dimensions* of quantities are denoted by the indices or exponents which belong to them; and when the sums of such exponents, either expressed or understood, are equal in all the terms of any expression, those terms are said to be *Homogeneous*.

Thus, the degrees or orders of a, $(b+x)^2$ and $(a-x)^m$ are 1, 2 and m respectively; and the terms of

$$x^3 + ax^2 + a^2x + a^3$$

are homogeneous.

17. DEF. Quantities are said to be *arranged* according to the dimensions of any letter involved in them, when the indices of that letter occur in the order of their magnitudes, either increasing or decreasing.

Thus, the terms of $a^2 - ax + x^2$ are arranged according to *descending* powers of a, and *ascending* powers of x.

18. DEF. One Algebraical Expression is said to be of a *higher* or *lower* order than another, according as the

letter which characterises its terms has a *larger* or *smaller* index.

Thus, $x^3 - ax^2 + a^2x - a^3$ is of a higher order than $x^2 + ax + a^2$.

19. Def. In addition to the Arithmetical Signs of *Equality* and *Proportionality*, namely, $=$ and $: :: :$, it is found convenient to adopt certain abbreviations for words of frequent occurrence: as $>$ is equivalent to *greater than*, $<$ to *less than*, $\because$ to *since* or *because*, and $\therefore$ to *therefore* or *consequently*.

Thus, $ax - b = cx + d$, implying that the quantities on each side of the sign $=$, are equal to one another, the whole is termed an *Equality* or *Equation*.

Also, $a : b :: c : d$, or $a : b = c : d$ denotes that a is the same multiple, part or parts of b, that c is of d, and has the same signification as $\frac{a}{b} = \frac{c}{d}$.

Again, $ax - b > cx + d$, and $ax - b < cx + d$, are termed *greater* and *less Inequalities*.

20. In the rudiments of this science, certain other terms are frequently used, which in a great degree explain themselves.

Like Quantities, as, a, $2a$: $4ab$, $7ab$, $9ab$.

Unlike Quantities, as, a, b: $3x$, $5xy$, $7cz$.

Monomials, as, a, ab, cdx.

Binomials, as, $a + b$, $a - bx$, $5a + 7x$.

Trinomials, as, $a + b + c$, $x^2 - px + q$.

Multinomials, as, $a + bx + cx^2 + dx^3 +$ &c.

21. We may illustrate the definitions already given, by the following examples, in which a, b, c, d, e, &c. are supposed to represent the natural numbers 1, 2, 3, 4, 5, &c.

Thus, $a + b + c - d = 1 + 2 + 3 - 4 = 2$.

$ab + ac - bc + cd = 1.2 + 1.3 - 2.3 + 3.4 = 2 + 3 - 6 +$

$$(a + c)(d - b) = (1 + 3)(4 - 2) = 4.2 = 8.$$

$$\frac{a - b + c}{b + d - e} = \frac{1 - 2 + 3}{2 + 4 - 5} = \frac{2}{1} = 2.$$

$$\frac{ab + de}{ac + cd} = \frac{1.2 + 4.5}{1.3 + 3.4} = \frac{2 + 20}{3 + 12} = \frac{22}{15} = 1\tfrac{7}{15}.$$

$$(ac + b^2)^2 = (1.3 + 2^2)^2 = (3 + 4)^2 = 7^2 = 49.$$

$$\{(a + b)(e - c)\}^3 = \{(1 + 2)(5 - 3)\}^3 = (3.2)^3 = 6^3 = 216.$$

$$\left(\frac{d - a}{c - b}\right)^4 = \left(\frac{4 - 1}{3 - 2}\right)^4 = \left(\frac{3}{1}\right)^4 = 3^4 = 81.$$

$$\sqrt{abcd + a^4} = \sqrt{1.2.3.4 + 1^4} = \sqrt{24 + 1} = \sqrt{25} = 5.$$

$$\left(\frac{a^2 + bc + de}{ab + bc}\right)^{\frac{1}{3}} = \left(\frac{1^2 + 2.3 + 4.5}{1.2 + 2.3}\right)^{\frac{1}{3}} = \left(\frac{27}{8}\right)^{\frac{1}{3}} = \frac{3}{2} = 1\tfrac{1}{2}.$$

In each of these instances, the monomials concerned are connected together by one of the signs + and − ; but it may be observed that in *Arithmetical Algebra* these signs do not in any way affect their *absolute* magnitudes, and that the terms *positive* and *negative* are applied to them merely in reference to other quantities, to which they are to be *added*, or from which they are to be *subtracted*, so that in consequence of quantity being in *general* increased by the former operation and diminished by the latter, *positive* and *negative* magnitudes are sometimes considered to be respectively *greater* and *less* than *zero* or 0.

CHAPTER II.

THE FUNDAMENTAL OPERATIONS ON ALGEBRAICAL QUANTITIES.

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22. The Fundamental Operations in Algebra, though analogous to those in Arithmetic, possess a character peculiar to themselves, in consequence of the generality of the Symbols employed: thus,

Addition and *Subtraction* are the *combining* or *incorporating* into *one* expression, *two* or *more* others which are *like*, according to the operations indicated by their respective Algebraical Signs, and the *placing* those that are *unlike*, one after another in a line, with their proper signs prefixed.

Multiplication is the *incorporating* two or more quantities, either simple or compound, indicating a product, in such a manner as to exhibit the result by a connected series of *simple* terms: and *Division* is merely the reverse of the last operation, determining from a series of *simple* terms considered as a product, *one* of the factors which have produced it, by means of the *other*, supposed to be *given*.

Involution is a repetition of the multiplication just referred to, when the factors are supposed to be *equal:* and *Evolution* is only the method of ascending back again to the source from which the result of the last mentioned operation has arisen.

From this account it will appear that all these operations in a great measure depend upon the Definitions laid down

in the preceding chapter, and will consequently be founded upon reasoning identical with that of Arithmetic, as far as the two sciences have their principles in common. For our present purpose, it will be sufficient to look upon the Rules of Arithmetic as our guide, and to notice, in the progress of the work, any case that may occur, wherein a deviation from these rules, as applied to symbols used in an extended sense, may seem to require it.

I. ADDITION.

23. The sum of any number of *like* quantities with the *same* sign, will be found by taking the sum of their numerical coefficients, prefixing it to the common letters, and retaining the sign common to them all.

(1)	$3ax$	(2)	$-2by$	(3)	$5a-6b$
	$4ax$		$-4by$		$8a-4b$
	$7ax$		$-6by$		$11a-23b$
	$14ax$		$-12by$		$24a-33b$

In (1), it is evident that by adding together $3ax$, $4ax$ and $7ax$, we shall have the sum equal to *three* times the quantity ax, together with *four* times that quantity, and *seven* times the same quantity: that is, the sum will be *fourteen* times the quantity ax, or $14ax$.

In (2), the same considerations prove the sum of $-2by$, $-4by$ and $-6by$ to be $-12by$, all the quantities being merely symbolical as they stand by themselves.

In (3), we have first found the sum of $5a$, $8a$, and $11a$ to be $24a$, and from this is to be *subtracted* the sum of $6b$, $4b$ and $23b$ which is $33b$, so that the proper result is $24a-33b$.

24. The sum of any number of *like* quantities with *different* signs, will be obtained by taking the excess of the sum of those with one of the signs, above the sum of those with the other: and by prefixing the sign of the greater sum, agreeably to the views of common arithmetic.

$$
\begin{array}{lr}
(4) & 4ax \\
& -5ax \\
& -2ax \\
& 8ax \\
\hline
& 5ax \\
\hline
\end{array}
\qquad
\begin{array}{lr}
(5) & 3by \\
& -7by \\
& 2by \\
& -4by \\
\hline
& -6by \\
\hline
\end{array}
\qquad
\begin{array}{lr}
(6) & 13x^4 - \ \ ax^3 \\
& -10x^4 + 7ax^3 \\
& 3x^4 - 6ax^3 \\
& 7x^4 - 4ax^3 \\
\hline
& 13x^4 - 4ax^3 \\
\hline
\end{array}
$$

In (4), we observe that $4ax$ and $8ax$ being *positive*, together make $12ax$; also, the *negative* quantities $-5ax$ and $-2ax$ amount to $-7ax$; and therefore the incorporated expression will be equivalent to $12ax - 7ax$ or $5ax$.

In (5), the *positive* quantities amount to $5by$, and the *negative* to $-11by$; and this shews that there is an excess of the quantities to be *subtracted* above those to be *added*, equal to $6by$; and the sum is therefore equivalent to $-6by$.

In (6), the *positive* quantities of the first vertical row exceed the *negative* by $13x^4$, and the *negative* quantities of the second exceed the *positive* by $4ax^3$; so that the algebraical sum is properly expressed by $13x^4 - 4ax^3$.

25. When the quantities which are *like*, are not arranged in the same order, a similar method is adopted; but it will generally be convenient to make such an arrangement before the operation is performed, in order to facilitate the process.

$$
\begin{array}{lr}
(7) & 5a - 10b + \ \ 3c \\
& 2b - \ \ 3a - \ \ 7c \\
& -\ \ 5c + \ \ 8b - 15a \\
\hline
& -13a + \ \ 0 \ \ - \ \ 9c \\
\hline
\end{array}
\qquad
\begin{array}{lr}
(8) & 5a - 10b + 3c \\
& -\ \ 3a + \ \ 2b - 7c \\
& -15a + \ \ 8b - 5c \\
\hline
& -13a + \ \ 0 \ \ - 9c \\
\hline
\end{array}
$$

In the latter of these forms, the like quantities of the former have been arranged in the same order: and it is seen that b disappears in the sum, inasmuch as $+10b$ is neutralized by $-10b$, from the nature of the operation; and the result may be written $-13a - 9c$.

26. The preceding articles being applicable only to *like* quantities, having either the *same* or *different* signs, it remains

that quantities, differing in their symbolical representation, can be added together only by connecting them with each other, by means of their proper individual signs: thus,

(9) The sum of $5ax$, $7by$, $-8cz$ and $14d$, must be expressed by $5ax+7by-8cz+14d$, agreeably to the observation made in article (22).

Examples for Practice.

(1) Find the sum of $4ax+3by$, $5ax+8by$, $8ax+6by$ and $20ax+by$.

Answer: $37ax+18by$.

(2) Add together $10cz-2ax^2$, $15cz-3ax^2$, $24cz-9ax^2$ and $3cz-8ax^2$.

Answer: $52cz-22ax^2$.

(3) Combine into one sum, $3x^2y^2-10y^4$, $-x^2y^2+5y^4$, $8x^2y^2-6y^4$ and $4x^2y^2+2y^4$

Answer: $14x^2y^2-9y^4$.

(4) Express in its simplest form, the sum of

$12a+5c+17d+13b$, $8a+12b+15d+8c$,

$11c+15a+23b+10d$ and $4d+3a+20b+18c$.

Answer: $38a+68b+42c+46d$.

(5) Add together $5a+3b-4c$, $2a-5b+6c+2d$, $a-4b-2c+3e$ and $7a+4b-3c-6e$.

Answer: $15a-2b-3c+2d-3e$.

(6) Find the sum of $3a^2+2ab+4b^2$, $5a^2-8ab+6b^2$, $-4a^2+5ab-b^2$, $18a^2-20ab-19b^2$ and $14a^2-3ab+20b^2$.

Answer: $36a^2-24ab+10b^2$.

(7) Simplify as much as possible, the sum of

$4x^3-5ax^2+6a^2x-5a^3$, $3x^3+4ax^2+2a^2x+6a^3$,

$-17x^3+19ax^2-15a^2x+8a^3$, $13ax^2-27a^2x+18a^3$

and $12x^3+3a^2x-20a^3$.

Answer: $2x^3+31ax^2-31a^2x+7a^3$.

(8) Incorporate as much as possible, the sum of

$5xy - 7ez + 18ax - 14by$, $3xy - 5cd + 11eg + 14ez$, $13ax + 20eg - 35cd + 18$ and $25xy - 15eg + 9by - 12ax$.

Answer: $33xy + 7ez + 19ax - 5by - 40cd + 16eg + 18$.

(9) Required the sum of the expressions,

$10a^2b - 12a^3bc - 15b^2c^4 + 10$, $-4a^2b + 8a^3bc - 10b^2c^4 - 4$, $-3a^2b - 3a^3bc + 20b^2c^4 - 3$ and $2a^2b + 12a^3bc + 5b^2c^4 + 2$.

Answer: $5a^2b + 5a^3bc + 5$.

(10) Add together the following five quantities:

$a + b + c + d$, $a + b + c - d$, $a + b - c + d$, $a - b + c + d$ and $-a + b + c + d$.

Answer: $3a + 3b + 3c + 3d$.

II. SUBTRACTION.

27. Subtraction being the reverse of Addition, it is evident that those quantities which are to be combined with others by the operation of Subtraction must be supposed to be affected with signs contrary to what they would have been by the operation of Addition: and this amounts to the same thing as "Changing the signs of the quantities to be subtracted, or conceiving them to be changed, and then combining them with the others by the operation of Addition."

(1)		(2)		(3)	
	$10c$		$-5y$		$4ax + 2by$
	$7c$		$+8y$		$2ax + by$
	$3c$		$-13y$		$2ax + by$

In (1), we observe that $10c = 7c + 3c$, from which if $7c$ be subtracted, or taken away, there remains $3c$: and this is the same as the algebraical sum of $10c$ and $-7c$, or a consequence of article (24).

In (2), $8y$ is to be subtracted, or affected with the negative sign, and this together with $-5y$ which has already the same sign, will manifestly give the symbolical result $-13y$.

In (3), we have by the preceding cases,

$$4ax - 2ax = 2ax, \text{ and } 2by - by = by,$$

so that the remainder is $2ax + by$.

28. We will further confirm the principle of the rule laid down in the last article, by the consideration of the following instances.

(1) To subtract $b - c$ from a, we observe that if b alone were taken from a, the remainder would be expressed by $a - b$: but inasmuch as we have by this process taken away from a, a quantity too large by c, it follows that the remainder will be too small by the same quantity; and therefore the proper result will be $a - b$ increased by c: that is, $a - (b - c)$ is equivalent to $a - b + c$.

(2) In subtracting $a - b$ from a, it is evident that the remainder will not be affected by increasing the *Minuend* and *Subtrahend* by the same quantity: and consequently we shall have $a - (a - b)$ equivalent to

$$a + b - (a - b + b) = a + b - a = b.$$

(3) If we wish to subtract $a + b$ from a, it may be observed that the result will be the same when both these quantities are diminished by b, and thus we shall have

$$a - (a + b) = a - b - (a + b - b) = a - b - a = -b,$$

for the remainder.

(4) Of the last instance, the result belongs entirely to Symbolical Algebra; as, for example, if we attempt to subtract 5 from 3, it is clear that the remainder which, according to these principles, would be -2, can admit of no explanation consistent with the views of common Arithmetic, but may be accepted in accordance with the observations made at the end of the preceding chapter.

Examples for Practice.

(1) Subtract $2a^2 + 3bc$ from $5a^2 + 7bc$.

Answer: $3a^2 + 4bc$.

(2) Find the excess of

$$6a^2 + 12ab + 19b^2 + c^2$$

above $4a^2 + \ 8ab + 13b^2$.

Answer: $2a^2 + 4ab + 6b^2 + c^2$.

(3) From $11a^2 + 12ab + 4b^2 + 7ac + 9c^2$

take $7a^2 + \ 6ab + b^2 + 2ac + 4c^2$.

Answer: $4a^2 + 6ab + 3b^2 + 5ac + 5c^2$.

(4) Required the excess of

$$5a^2 + 4ab - 3ac + bc - 3c^2$$

above $3a^2 + 3ab + 3bc - 2c^2$.

Answer: $2a^2 - 3ac + ab - 2bc - c^2$.

(5) Find how much

$$12x + 6a - 4b - 12c - 7e - 5f$$

exceeds $2x - 3a + 4b - 5c + 6d - 7e$.

Answer: $10x + 9a - 8b - 7c - 6d - 5f$.

(6) Subtract $18ax^3 + 20a^2x^2 - 24a^3x - \ 7a^4$

from $28ax^3 - 16a^2x^2 + 25a^3x - 13a^4$.

Answer: $10ax^3 - 36a^2x^2 + 49a^3x - 6a^4$.

(7) Take $a^2xy + 3bx^2y - 13cxy^2 + 20y^5$

from $8a^2xy - 5bx^2y + 17cxy^2 - \ 9y^5$.

Answer: $7a^2xy - 8bx^2y + 30cxy^2 - 29y^5$.

(8) From $6x^3y - 10x^2y^2 + 13xy^3 - 19y^4$

take $-5x^3y + \ 2x^2y^2 - \ 3xy^3 + \ 2y^4$.

Answer: $11x^3y - 12x^2y^2 + 16xy^3 - 21y^4$.

(9) Subtract $x^3 - px^2 + qx - r$ from $x^3 - ax^2 + bx - c$.

Answer: $px^2 - ax^2 + bx - qx + r - c$.

(10) From $-17x^3 + 9ax^2 - 7a^2x + 15a^3$

take $-19x^3 + 9ax^2 - 9a^2x + 17a^3$.

Answer: $2x^3 + 2a^2x - 2a^3$.

On the Effects of the Bracket or Vinculum in Addition and Subtraction.

29. Whenever an algebraical quantity consists of two or more terms, it has been stated in article (6), that it is frequently inclosed in a *Bracket*, or connected by a *Vinculum*, in order to keep it distinct from others with which it may be combined by the operations of Arithmetic: and we will now endeavour to point out the influence of a bracket in the two operations which we have already considered.

(1) If a set of quantities inclosed in a bracket be supposed to be combined with one or more others by the operation of *Addition*, or by means of the sign +; it is evident that the bracket can have no effect upon the result, and may therefore be retained or not, at pleasure.

Thus, $a + (b + c)$ is manifestly equivalent to $a + b + c$: for it makes no difference whether b and c be first added together, and the sum be then added to a; or the sum of the three quantities a, b, c be taken at once.

Again, $x - y + (b - z)$ will amount to the same thing as $x - y + b - z$: for it is clearly immaterial whether $b - z$ be added to $x - y$ at once, or b be added to it at first, and from the result, z be then subtracted.

Moreover it appears that $x - y + b - z$, $x + b - y - z$, $x - z - y + b$ and $-y - z + b + x$ are all equivalent expressions: and a bracket may at any time be introduced, whenever it is found expedient to keep one part of an expression detached from the rest.

(2) If a quantity included in a bracket, be combined with others by the operation of *Subtraction*, or by means of the sign $-$; the rule laid down in article (27) shews that the signs of the terms of this quantity must be changed, whenever the bracket is removed.

Thus, $a-(b+c)$ is equivalent to $a-b-c$: because it is evidently of no importance whether b be first subtracted from a, and c be then taken from the remainder, or the sum of b and c be subtracted from a at once.

This may also be made to appear symbolically as follows:

$$a-(b+c) \text{ is clearly equivalent to}$$
$$a-b-(b-b+c)=a-b-(+c)=a-b-c.$$

Again, $a-x-(b-y)$ is, by the same method of reasoning, shewn to be equivalent to $a-x-b+y$: for by the nature of the operation indicated by the sign $-$, we have

$$a-x-(b-y) \text{ equivalent to}$$
$$a-x-b-(b-b-y)=a-x-b-(-y)$$
$$=a-x-b+y.$$

Conversely, a Bracket with a negative sign preceding it, may be introduced into any compound Algebraical expression, provided the signs of all the symbols comprised in it be changed: thus,

$$a-x-b+y \text{ is equivalent to } a-x-(b-y),$$
$$\text{or } a-(x+b-y), \text{ or } a+y-(b+x), \text{ or } a+y+(-b-x).$$

(3) Similar considerations will enable us to dispense with the brackets without altering the values of the expressions, when one or more such brackets are included within another.

Thus, $a-\{b-(c+d)\}$ is manifestly equivalent to

$$a-\{b-c-d\} \text{ equivalent to } a-b+c+d:$$

also, $$a - \{a + b - [a + b - c - (a - b + c)]\}$$
$$= a - \{a + b - [a + b - c - a + b - c]\}$$
$$= a - \{a + b - [2b - 2c]\} = a - \{a + b - 2b + 2c\}$$
$$= a - \{a - b + 2c\} = a - a + b - 2c = b - 2c.$$

Examples for Practice.

(1) Simplify as much as possible, the expression
$$(1 - 2x + 3x^2) + (3 + 2x - x^2).$$
Answer: $4 + 2x^2$.

(2) Reduce to its simplest form, the expression
$$5a - 4b + 3c + (-3a + 2b - c).$$
Answer: $2a - 2b + 2c$.

(3) Exhibit $a - (b - c) + b - (a - c) + c - (a - b)$ in its simplest form.

Answer: $-a + b + 3c$, or $-(a - b - 3c)$.

III. Multiplication.

30. The product of two quantities being arithmetically equivalent to the sum of the products arising from multiplying either of them, by the parts of which the other is made up, it will be necessary to consider here the symbolical results arising from this incorporation of quantities affected with the signs + and −, in accordance with the notation explained in article (8).

(1) Since a product has a *symmetrical* relation to its factors, it follows that $a \times b = b \times a$, or $ab = ba$: and that abc, acb, bac are equivalent expressions.

(2) By the nature of the operation implied, we have
$$a \times (b + c), \text{ or } a(b + c) = ab + ac,$$
and $$a \times (b - c), \text{ or } a(b - c) = ab - ac.$$

(3) The product of $a+b$ and $c+d$, will evidently be equal to the product of a and $c+d$, *increased* by the product of b and $c+d$: that is, we shall have

$$(a+b)(c+d) = a(c+d) + b(c+d)$$

$$= ac + ad + bc + bd.$$

(4) The product of $a-b$ and $c-d$, will manifestly be $a-b$ taken c times, or $(a-b)c = ac - bc$, *diminished* by $a-b$ taken d times, or by $(a-b)d = ad - bd$: that is,

$$(a-b)(c-d) = (ac - bc) - (ad - bd)$$

$$= ac - bc - ad + bd,$$

by the last article.

In this instance, when $a-b$ and $c-d$ are *negative* or *symbolical*, the product must still be of the same form: for, otherwise the result of this Algebraical Process would be restricted by the specific values and natures of the symbols employed, contrary to the *assumed* generality of its principles.

(5) If we examine the product of $a-b$ and $c-d$, which has been proved to be $ac - bc - ad + bd$, and consider in what manner each of its terms originates, we shall find that

$$(+a) \times (+c) = +ac, \quad (-b) \times (+c) = -bc,$$

$$(+a) \times (-d) = -ad, \quad (-b) \times (-d) = +bd:$$

and these equalities expressed in words, furnish us with the following general direction, which is called the *Rule of Signs*.

"The product of two simple Algebraical Quantities is preceded by the *positive* or *negative* sign, according as the signs of the factors are the *same* or *different*."

Ex. 1. Multiply $x^2 + ax + a^2$ by $x + a$.

Here, adopting a form analogous to the corresponding one of Arithmetic, and beginning with the symbols on the *left* hand, we have

$$\begin{array}{llll}
x^2 + ax + a^2 & & & \\
x + a & & & \\
\hline
x^3 + ax^2 + a^2x & & = \text{the product by } +x: \\
\quad\quad ax^2 + a^2x + a^3 & & = \text{the product by } +a: \\
\hline
x^3 + 2ax^2 + 2a^2x + x^3 & & = \text{the product by } x+a. \\
\hline
\end{array}$$

Ex. 2. Multiply $3x^2 - 2xy - y^2$ by $2x - 4y$.

Here,
$$\begin{array}{ll}
3x^2 - 2xy - y^2 & \\
2x - 4y & \\
\hline
6x^3 - 4x^2y - 2xy^2 & = \text{the product by } +2x: \\
\quad\quad - 12x^2y + 8xy^2 + 4y^3 & = \text{the product by } -4y: \\
\hline
6x^3 - 16x^2y + 6xy^2 + 4y^3 & = \text{the product by } 2x - 4y. \\
\hline
\end{array}$$

Ex. 3. Find the product of $x^2 + 2x + 1$ and $x^2 - 2x + 3$: also, of $a - b + c$ and $a + b - c$.

Here,
$$\begin{array}{l}
x^2 + 2x + 1 \\
x^2 - 2x + 3 \\
\hline
x^4 + 2x^3 + x^2 \\
\quad\quad - 2x^3 - 4x^2 - 2x \\
\quad\quad\quad\quad 3x^2 + 6x + 3 \\
\hline
x^4 + 4x + 3 \\
\hline
\end{array}
\qquad\qquad
\begin{array}{l}
a - b + c \\
a + b - c \\
\hline
a^2 - ab + ac \\
\quad\quad ab - b^2 + bc \\
\quad\quad\quad\quad - ac + bc - c^2 \\
\hline
a^2 - b^2 + 2bc - c^2 \\
\hline
\end{array}$$

Ex. 4. Express by means of simple terms, the continued product $(1 + x) \times (1 - x + x^2 - x^3) \times (1 + x^4)$.

Here,
$$
\begin{array}{l}
1 - x + x^2 - x^3 \\
1 + x \\
\hline
1 - x + x^2 - x^3 \\
\quad\; x - x^2 + x^3 - x^4 \\
\hline
1 - x^4 = \text{the product of the first two factors:} \\
1 + x^4 \\
\hline
1 - x^4 \\
\quad\; x^4 - x^8 \\
\hline
1 - x^8 = \text{the continued product required.}
\end{array}
$$

31. When the terms of the factors are characterised by different powers of the same letter, it will generally be found convenient to arrange them according to their dimensions, as has been done in the preceding examples: and whenever two or more terms of a product comprise a letter or letters in common, it is usual to abbreviate the expression of it by means of a bracket, as will be seen in the following instances.

Ex. 1. Multiply $x^2 - px + q$ by $x - a$.

Here,
$$
\begin{array}{l}
x^2 - px + q \\
x - a \\
\hline
x^3 - px^2 + qx \\
\quad - ax^2 + apx - aq \\
\hline
x^3 - (a + p)\,x^2 + (ap + q)\,x - aq \\
\hline
\end{array}
$$

Ex. 2. Find the product of $x^4 - (n - 1)\,a^2x^2 + a^4$ and $x^2 - a^2$.

$$\begin{array}{lllll}
\text{Here,} & x^4 - (n-1)\,a^2x^2 + a^4 \\
& x^2 - a^2 \\
\hline
& x^6 - (n-1)\,a^2x^4 + a^4x^2 \\
& \quad - \qquad\quad a^2x^4 + (n-1)\,a^4x^2 - a^6 \\
\hline
& x^6 - na^2x^4 + na^4x^2 - a^6 \\
\hline
\end{array}$$

Ex. 3. Required the continued product of $x+a$, $x-b$ and $x+c$.

$$\begin{array}{ll}
\text{Here,} & x + a \\
& x - b \\
\hline
& x^2 + ax \\
& \quad - bx - ab \\
\hline
& x^2 + (a-b)\,x - ab \\
& x + c \\
\hline
& x^3 + (a-b)\,x^2 - abx \\
& \qquad\quad cx^2 \quad + (ac - bc)\,x - abc \\
\hline
& x^3 + (a-b+c)\,x^2 - (ab - ac + bc)\,x - abc. \\
\hline
\end{array}$$

Examples for Practice.

(1) Find the products of $3x+2y$ and $2x+3y$; also, of $2ab-3b^2$ and $3ab+4b^2$.

Answers: $6x^2+13xy+6y^2$, and $6a^2b^2-ab^3-12b^4$.

(2) Required the products of x^2+xy+y^2 and x^2-xy+y^2; also, of $9a^2+3ax+x^2$ and $9a^2-3ax+x^2$.

Answers: $x^4+x^2y^2+y^4$, and $81a^4+9a^2x^2+x^4$.

(3) Multiply $27x^3+9x^2y+3xy^2+y^3$ by $3x-y$, and $a^4-2a^3b+4a^2b^2-8ab^3+16b^4$ by $a+2b$.

Answers: $81x^4-y^4$, and a^5+32b^5.

(4) Multiply $1 + x + x^4 + x^5$ by $1 - x + x^2 - x^3$, and $x^4 - x^3 + x^2 - x + 1$ by $x^2 + x - 1$.

Answers: $1 - x^8$, and $x^6 - x^4 + x^3 - x^2 + 2x - 1$.

(5) Multiply $x^3 - x^2y + xy^2 - y^3$ by $x^3 + x^2y + xy^2 + y^3$, and $x^3 + 3ax^2 + 3a^2x + a^3$ by $x^3 - 3ax^2 + 3a^2x - a^3$.

Answers: $x^6 + x^4y^2 - x^2y^4 - y^6$, and $x^6 - 3a^2x^4 + 3a^4x^2 - a^6$.

(6) Multiply $a^{m-1} + b^{m-1}$ by $a^{n+1} - b^{n+1}$, and $x^{m+n} + y^{m-n}$ by $x^{m-n} - y^{m+n}$.

Answers: $a^{m+n} + a^{n+1}b^{m-1} - a^{m-1}b^{n+1} - b^{m+n}$,
and $x^{2m} - (xy)^{m+n} + (xy)^{m-n} - y^{2m}$.

(7) Multiply $a^2 + b^2 + c^2 - ab - ac - bc$ by $a + b + c$.

Answer: $a^3 + b^3 + c^3 - 3abc$.

(8) Multiply $1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + 7x^6 - 8x^7$ by $1 + 2x + x^2$.

Answer: $1 - 9x^8 - 8x^9$.

(9) Multiply $x^2 + ax + b$ by $x^2 - ax + c$.

Answer: $x^4 - (a^2 - b - c)x^2 - (b - c)ax + bc$.

(10) Multiply $a + x + x^2 + x^3 + x^4$ by $a - x$.

Answer: $a^2 + (a - 1)x^2 + (a - 1)x^3 + (a - 1)x^4 - x^5$.

(11) Multiply $x^2 + (n + 1)ax - a^2$ by $x^2 - (n - 1)ax + a^2$.

Answer: $x^4 + 2ax^3 + (n^2 - 1)a^2x^2 + 2na^2x - a^4$.

(12) Prove that the continued product of
$x - 3$, $x + 3$, $x - 4$ and $x + 4$, is $x^4 - 25x^2 + 144$.

(13) Shew that

$(a^2 + ab + b^2)(a^3 - a^2b + b^3)(a - b) = a^6 - a^5b + a^2b^4 - b^6$.

(14) The continued product of $a + b$, $a - b$, $a^2 + ab + b^2$ and $a^2 - ab + b^2$, is $a^6 - b^6$.

(15) The continued product of $x-a$, $x-b$, $x-c$ and $x-d$, is

$$x^4-(a+b+c+d)x^3+(ab+ac+ad+bc+bd+cd)x^2$$
$$-(abc+abd+acd+bcd)x+abcd.$$

(16) Simplify as much as possible, the expression,
$(x+a)(x+b)(x+c)-(a+b+c)(x+a)(x+b)+(a^2+ab+b^2)(x+a)$.

Answer: x^3+a^3.

(17) Prove the identity expressed by the equation,

$$(a-b)(a+b-c)+(b-c)(b+c-a)+(c-a)(a+c-b)=0.$$

(18) If $a=n^2-1$, $b=2n$ and $c=n^2+1$: verify the equality $(a+b+c)(a+b-c)(a+c-b)(b+c-a)=4a^2b^2$.

IV. Division.

32. The quotient of one algebraical quantity by another, being that quantity which multiplied by the latter produces the former, will be obtained by reversing the last operation conformably to the principles already explained.

In this process, the *Rule of Signs* previously established still holds good: thus,

since $(+a)\times(+b)=+ab$, we have $\dfrac{+ab}{+b}=+a$:

similarly,

$(+a)\times(-b)=-ab$, gives $\dfrac{-ab}{-b}=+a$, and $\dfrac{-ab}{+a}=-b$,

and from $(-a)\times(-b)=+ab$, we obtain $\dfrac{+ab}{-a}=-b$.

33. When the divisor is a simple quantity, the division is indicated according to article (9), and the result is then reduced to its simplest form: thus,

the quotient of $6a^2c$ by $2a$, is $\dfrac{6a^2c}{2a}=3ac$:

the quotient of $9a^2bc - 12ab^2c + 15abc^2$ by $3ab$, is

$$\frac{9a^2bc}{3ab} - \frac{12ab^2c}{3ab} + \frac{15abc^2}{3ab} = 3ac - 4bc + 5c^2:$$

and both these results are easily verified.

34. When the divisor is a compound quantity, the operation is conducted in the form of *Long Division* in Arithmetic: and the circumstance most worthy of attention is the arrangement of the terms of the divisor and dividend, according to the dimensions of some common symbol.

Ex. 1. Divide $a^2 + 6ab + 8b^2$ by $a + 4b$.

Here,

$$\begin{array}{l} a + 4b)\ a^2 + 6ab + 8b^2\ (a + 2b \\ \qquad\quad\ \underline{a^2 + 4ab\qquad} \\ \qquad\qquad\quad 2ab + 8b^2 \\ \qquad\qquad\quad \underline{2ab + 8b^2} \end{array}$$

so that the required quotient is $a + 2b$; also, the terms of the divisor and dividend are at *every* step arranged according to the dimensions of the symbol a, and the operation is performed exactly as in common Arithmetic.

Ex. 2. Find the quotient of $a^4 + 4a^2b^2 + 16b^4$ by $a^2 - 2ab + 4b^2$.

Here,

$$\begin{array}{l} a^2 - 2ab + 4b^2)\ a^4 + 4a^2b^2 + 16b^4\ (a^2 + 2ab + 4b^2 \\ \qquad\qquad\qquad \underline{a^4 - 2a^3b + 4a^2b^2} \\ \qquad\qquad\qquad\qquad 2a^3b + 16b^4 \\ \qquad\qquad\qquad\qquad \underline{2a^3b - 4a^2b^2 + 8ab^3} \\ \qquad\qquad\qquad\qquad 4a^2b^2 - 8ab^3 + 16b^4 \\ \qquad\qquad\qquad\qquad \underline{4a^2b^2 - 8ab^3 + 16b^4} \end{array}$$

that is, the quotient is $a^2 + 2ab + 4b^2$, as may easily be verified; and it is observed that in every stage of the proceeding, the terms involving the highest powers of a have been placed foremost.

Ex. 3. Divide $x^3 - px^2 + qx - r$ by $x - a$.

Here,

$$
\begin{array}{l}
x - a)\, x^3 - px^2 + qx - r\ (x^2 + (a-p)\,x + (a^2 - pa + q) \\
\qquad\quad \underline{x^3 - ax^2\qquad\qquad} \\
\qquad\qquad (a-p)\,x^2 + qx \\
\qquad\qquad \underline{(a-p)\,x^2 - (a^2 - pa)\,x} \\
\qquad\qquad\qquad\quad (a^2 - pa + q)\,x - r \\
\qquad\qquad\qquad\quad \underline{(a^2 - pa + q)\,x - (a^3 - pa^2 + qa)} \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{a^3 - pa^2 + qa - r}
\end{array}
$$

the steps of the operation being effected as in the preceding examples, and the remainder being $a^3 - pa^2 + qa - r$, which, it may be observed, is of the same *form* as the dividend with a in the place of x.

Ex. 4. Let it be required to find the result of the division of 1 by $1 + x$.

Here,

$$
\begin{array}{l}
1 + x)\, 1 \qquad (1 - x + x^2 - x^3 + \&c. \\
\qquad\quad \underline{1 + x} \\
\qquad\quad\ - x \\
\qquad\quad\ \underline{- x - x^2} \\
\qquad\qquad\qquad x^2 \\
\qquad\qquad\qquad \underline{x^2 + x^3} \\
\qquad\qquad\qquad\quad - x^3 \\
\qquad\qquad\qquad\quad \underline{- x^3 - x^4} \\
\qquad\qquad\qquad\qquad\quad x^4
\end{array}
$$

and in this we see that the index of x in the remainder is always equal to the corresponding number of terms in the quotient: and it is manifest that the intended operation may be continued as far as we please.

The results in cases like the present, are generally termed *Infinite Series*, and written in the following form:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \&c. \textit{ in infinitum}:$$

but here an arithmetical equality is not implied, unless we retain the remainder at the point where the operation ceases: thus,

$$\frac{1}{1+x} = 1 - \frac{x}{1+x}, \quad \frac{1}{1+x} = 1 - x + \frac{x^2}{1+x},$$

$$\frac{1}{1+x} = 1 - x + x^2 - \frac{x^3}{1+x}, \quad \frac{1}{1+x} = 1 - x + x^2 - x^3 + \frac{x^4}{1+x},$$

&c. are all arithmetically true: whereas $1 - x + x^2 - x^3 +$ &c. *in infinitum*, must be looked upon, in Symbolical Algebra, as an expression *equivalent* to $\frac{1}{1+x}$, when used in its most general acceptation.

35. We will conclude with an additional article expressing a general property of numbers, the circumstances of which it will be convenient to retain in the memory.

To shew that $x^m - a^m$ is always divisible by $x - a$, whatever positive whole number m may be: we have

$$\begin{array}{l}
x - a)\ x^m - a^m\ (x^{m-1} + ax^{m-2} + a^2x^{m-3} + \&c. + a^{m-2}x + a^{m-1} \\
\qquad\ \underline{x^m - ax^{m-1}\qquad\quad} \\
\qquad\qquad\quad ax^{m-1} - a^m \\
\qquad\qquad\quad \underline{ax^{m-1} - a^2x^{m-2}\qquad\quad} \\
\qquad\qquad\qquad\qquad a^2x^{m-2} - a^m \\
\qquad\qquad\qquad\qquad \underline{a^2x^{m-2} - a^3x^{m-3}\qquad} \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{a^3x^{m-3} - a^m}
\end{array}$$

and, it is here observable that in the remainder, the index of x is *diminished* and that of a is *increased* by 1, in each succeeding step, and that the sum of the indices in every term is always equal to m: whence we shall at length obtain $a^{m-2}x^2 - a^m$ for a remainder: and continuing the division from this as before, we have

$$\begin{array}{l} a^{m-2}x^2 - a^m \\ \underline{a^{m-2}x^2 - a^{m-1}x} \\ \qquad\qquad a^{m-1}x - a^m \\ \qquad\qquad \underline{a^{m-1}x - a^m} \end{array}$$

from which it will follow that, $x^m - a^m$ is always exactly divisible by $x - a$, when m is a positive whole number: that is,

$$\frac{x^m - a^m}{x - a} = x^{m-1} + ax^{m-2} + a^2x^{m-3} + \&c. + a^{m-2}x + a^{m-1},$$

the number of terms of the latter member of the equality being m.

By assigning to m the values 2, 3, 4, 5, &c. in order, we have the following useful results:

$$\frac{x^2 - a^2}{x - a} = x + a:$$

$$\frac{x^3 - a^3}{x - a} = x^2 + ax + a^2:$$

$$\frac{x^4 - a^4}{x - a} = x^3 + ax^2 + a^2x + a^3:$$

$$\frac{x^5 - a^5}{x - a} = x^4 + ax^3 + a^2x^2 + a^3x + a^4: \ \&c.$$

The compound algebraical expression

$$x^{m-1} + ax^{m-2} + a^2x^{m-3} + \&c. + a^{m-2}x + a^{m-1},$$

may therefore be assumed to be symbolically equivalent to $\frac{x^m - a^m}{x - a}$, whatever be the relative values of x and a: and from this it follows that, when $x = a$, $a^{m-1} + a^{m-1} + a^{m-1} +$ &c. to m terms, or ma^{m-1} is equivalent to $\frac{a^m - a^m}{a - a}$, which in arithmetical symbols becomes $\frac{0}{0}$, wherein all traces of its nature and origin have disappeared.

EXAMPLES FOR PRACTICE.

(1) Divide

abx^3y^4 by bxy, and $5x^3y^3 - 40a^2x^2y^2 + 25a^4xy$ by $-5xy$.

Answers: ax^2y^3, and $-x^2y^2 + 8a^2xy - 5a^4$.

(2) Divide

$8a^2 + 26ab + 15b^2$ by $4a + 3b$,

and $x^4 + x^2y^2 + y^4$ by $x^2 - xy + y^2$.

Answers: $2a + 5b$, and $x^2 + xy + y^2$.

(3) Divide

$x^3 + 6x^2 + 9x + 4$ by $x + 4$, and $y^4 - 81$ by $y - 3$.

Answers: $x^2 + 2x + 1$, and $y^3 + 3y^2 + 9y + 27$.

(4) Divide

$x^4 - 9x^2 - 6xy - y^2$ by $x^2 + 3x + y$,

and $x^4 - 6x^3y + 9x^2y^2 - 4y^4$ by $x^2 - 3xy + 2y^2$.

Answers: $x^2 - 3x - y$, and $x^2 - 3xy - 2y^2$.

(5) Divide

$x^4 - 4x^3 + 6x^2 - 4x + 1$ by $x^2 - 2x + 1$,

and $x^4 - 2a^2x^2 + 16a^3x - 15a^4$ by $x^2 + 2ax - 3a^2$.

Answers: $x^2 - 2x + 1$, and $x^2 - 2ax + 5a^2$.

(6) Divide

$12a^4 - 26a^3b - 8a^2b^2 + 10ab^3 - 8b^4$ by $3a^2 - 2ab + b^2$,

and $256x^4 + 16x^2y^2 + y^4$ by $16x^2 + 4xy + y^2$.

Answers: $4a^2 - 6ab - 8b^2$, and $16x^2 - 4xy + y^2$.

(7) Divide

$a^5 + a^3b^2 + a^2b^3 + b^5$ by $a^2 - ab + b^2$,

and $x^4 - y^4$ by $x - y$.

Answers: $a^3 + a^2b + ab^2 + b^3$, and $x^3 + x^2y + xy^2 + y^3$.

(8) Divide

$a^8 + a^6b^2 + a^4b^4 + a^2b^6 + b^8$ by $a^4 + a^3b + a^2b^2 + ab^3 + b^4$.

Answer: $a^4 - a^3b + a^2b^2 - ab^3 + b^4$.

(9) Divide

$$a^{m+n} - a^m b^n + a^n b^m - b^{m+n} \text{ by } a^n - b^n,$$
$$\text{and } x^6 - 16a^3x^3 + 64a^6 \text{ by } x - 2a.$$

Answers: $a^m + b^m$, and

$$x^5 + 2ax^4 + 4a^2x^3 - 8a^3x^2 - 16a^4x - 32a^5.$$

(10) Divide

$$4a^2 + 6ab - 4ax + 9bx - 15x^2 \text{ by } 2a + 3x, \text{ and}$$
$$3a^2 + 8ab + 4b^2 + 10ac + 8bc + 3c^2 \text{ by } a + 2b + 3c.$$

Answers: $2a + 3b - 5x$, and $3a + 2b + c$.

(11) Divide

$$a^6 - b^6 \text{ by } a^3 - 2a^2b + 2ab^2 - b^3, \text{ and}$$
$$1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5 \text{ by } 1 - 3x + 3x^2 - x^3.$$

Answers: $a^3 + 2a^2b + 2ab^2 + b^3$, and $1 - 2x + x^2$.

(12) Divide

$$ax^3 - a^3x + x^m - a^2x^{m-2}, \text{ and}$$
$$x^3 - nax^2 + na^2x - a^3 \text{ by } x - a.$$

Answers: $ax^2 + a^2x + x^{m-1} + ax^{m-2}$,

and $x^2 - (n-1)ax + a^2$.

(13) The quotient of

$$1 - 9x^8 - 8x^9 \text{ by } 1 + 2x + x^2 \text{ is}$$
$$1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + 7x^6 - 8x^7:$$
$$\text{and of } 1 + x - 17x^8 + 15x^9 \text{ by } 1 - 2x + x^2 \text{ is}$$
$$1 + 3x + 5x^2 + 7x^3 + 9x^4 + 11x^5 + 13x^6 + 15x^7.$$

(14) The quotient of

$$a^2 + (a-1)x^2 + (a-1)x^3 + (a-1)x^4 - x^5 \text{ by } a - x \text{ is}$$
$$a + x + x^2 + x^3 + x^4, \text{ and of}$$
$$a + (a+b)x + (a+b+c)x^2 + (a+b+c)x^3 + (b+c)x^4 + cx^5$$
$$\text{by } a + bx + cx^2 \text{ is } 1 + x + x^2 + x^3.$$

(15) The quotient of

$$x^4 - (a+b+c+d)\,x^3 + (ab+ac+ad+bc+bd+cd)\,x^2 - (abc+abd+acd+bcd)\,x + abcd \text{ by } x^2 - (a+c)\,x + ac \text{ is}$$

$$x^2 - (b+d)\,x + bd.$$

(16) The quotient of

$$x^4 - px^3 + qx^2 - rx + s \text{ by } x - a \text{ is}$$

$$x^3 + (a-p)\,x^2 + (a^2 - pa + q)\,x + a^3 - pa^2 + qa - r,$$

with a remainder $a^4 - pa^3 + qa^2 - ra + s$.

(17) Prove the following Symbolical Equalities:

$$\frac{1}{1 - 2x + x^2} = 1 + 2x + 3x^2 + 4x^3 + \text{\&c. } \textit{in infinitum}:$$

$$\frac{1-x}{1 + x - x^2} = 1 - 2x + 3x^2 - 5x^3 + \text{\&c. } \textit{in infinitum}.$$

(18) Shew that $a^m + b^m$ is divisible by $a+b$ when m is odd, and not when it is even: also, that $a^m - b^m$ is divisible by $a+b$ when m is even, and not when it is odd. Give the results in the former case, when the values of m are 3 and 5, and in the latter, when they are 4 and 6.

V. Involution.

36. The Involution of Algebraical Quantities, will, from the nature of the operation, be effected by means of the rules already given for Multiplication.

Ex. 1. The square of $-xy$

$$= (-xy)\,(-xy) = x^2y^2:$$

the cube of $-xy$

$$= (-xy)\,(-xy)\,(-xy) = (x^2y^2)\,(-xy) = -x^3y^3:$$

the fourth power of $-xy$

$$= (-x^3y^3)\,(-xy) = x^4y^4: \text{ \&c.}$$

and from this instance it is clear that the signs of all *even* powers of a *negative* quantity will be *positive*, whilst those of its *odd* powers will be *negative*.

(9) Divide

$$a^{m+n} - a^m b^n + a^n b^m - b^{m+n} \text{ by } a^n - b^n,$$
$$\text{and } x^6 - 16a^3x^3 + 64a^6 \text{ by } x - 2a.$$

Answers: $a^m + b^m$, and

$$x^5 + 2ax^4 + 4a^2x^3 - 8a^3x^2 - 16a^4x - 32a^5.$$

(10) Divide

$$4a^2 + 6ab - 4ax + 9bx - 15x^2 \text{ by } 2a + 3x, \text{ and}$$
$$3a^2 + 8ab + 4b^2 + 10ac + 8bc + 3c^2 \text{ by } a + 2b + 3c.$$

Answers: $2a + 3b - 5x$, and $3a + 2b + c$.

(11) Divide

$$a^6 - b^6 \text{ by } a^3 - 2a^2b + 2ab^2 - b^3, \text{ and}$$
$$1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5 \text{ by } 1 - 3x + 3x^2 - x^3.$$

Answers: $a^3 + 2a^2b + 2ab^2 + b^3$, and $1 - 2x + x^2$.

(12) Divide

$$ax^3 - a^3x + x^m - a^2x^{m-2}, \text{ and}$$
$$x^3 - nax^2 + na^2x - a^3 \text{ by } x-a.$$

Answers: $ax^2 + a^2x + x^{m-1} + ax^{m-2}$,

and $x^2 - (n-1)\,ax + a^2$.

(13) The quotient of

$$1 - 9x^8 - 8x^9 \text{ by } 1 + 2x + x^2 \text{ is}$$
$$1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + 7x^6 - 8x^7:$$
$$\text{and of } 1 + x - 17x^8 + 15x^9 \text{ by } 1 - 2x + x^2 \text{ is}$$
$$1 + 3x + 5x^2 + 7x^3 + 9x^4 + 11x^5 + 13x^6 + 15x^7.$$

(14) The quotient of

$$a^2 + (a-1)x^2 + (a-1)x^3 + (a-1)x^4 - x^5 \text{ by } a - x \text{ is}$$
$$a + x + x^2 + x^3 + x^4, \text{ and of}$$
$$a + (a+b)x + (a+b+c)x^2 + (a+b+c)x^3 + (b+c)x^4 + cx^5$$
$$\text{by } a + bx + cx^2 \text{ is } 1 + x + x^2 + x^3.$$

(15) The quotient of

$$x^4 - (a + b + c + d)\,x^3 + (ab + ac + ad + bc + bd + cd)\,x^2$$
$$- (abc + abd + acd + bcd)\,x + abcd \text{ by } x^2 - (a + c)\,x + ac \text{ is}$$
$$x^2 - (b + d)\,x + bd.$$

(16) The quotient of

$$x^4 - px^3 + qx^2 - rx + s \text{ by } x - a \text{ is}$$
$$x^3 + (a - p)\,x^2 + (a^2 - pa + q)\,x + a^3 - pa^2 + qa - r,$$
with a remainder $a^4 - pa^3 + qa^2 - ra + s$.

(17) Prove the following Symbolical Equalities:

$$\frac{1}{1 - 2x + x^2} = 1 + 2x + 3x^2 + 4x^3 + \text{\&c. } \textit{in infinitum}:$$

$$\frac{1 - x}{1 + x - x^2} = 1 - 2x + 3x^2 - 5x^3 + \text{\&c. } \textit{in infinitum}.$$

(18) Shew that $a^m + b^m$ is divisible by $a + b$ when m is odd, and not when it is even: also, that $a^m - b^m$ is divisible by $a + b$ when m is even, and not when it is odd. Give the results in the former case, when the values of m are 3 and 5, and in the latter, when they are 4 and 6.

V. INVOLUTION.

36. The Involution of Algebraical Quantities, will, from the nature of the operation, be effected by means of the rules already given for Multiplication.

Ex. 1. The square of $-xy$

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the fourth power of $-xy$

$$= (-x^3y^3)\,(-xy) = x^4y^4: \text{ \&c.}$$

and from this instance it is clear that the signs of all *even* powers of a *negative* quantity will be *positive*, whilst those of its *odd* powers will be *negative*.

(9) The cube of $a^2 + 2a - 4$ is

$$a^6 + 6a^5 - 40a^3 + 96a - 64,$$

and of $a^2 - a - 1$ is $a^6 - 3a^5 + 5a^3 - 3a - 1$.

(10) The fourth power of $a^m - a^n$ is

$$a^{4m} - 4a^{3m+n} + 6a^{2m+2n} - 4a^{m+3n} + a^{4n},$$

and the fifth power of $1 + x$ is

$$1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5.$$

VI. EVOLUTION.

37. The Evolution of Algebraical Quantities is the reverse of Involution, and will therefore be effected by any method which will enable us to trace back the steps of the latter operation.

Ex. 1. The square root of a^2 is either $+a$ or $-a$, because the square of each of these quantities is a^2.

Ex. 2. The cube root of $-x^3y^6$ is $-xy^2$, since by the last article, we have $(-xy^2)^3 = -x^3y^6$.

Ex. 3. The m^{th} root of x^{mn} is x^n, because the m^{th} power of x^n is expressed by x^{mn}.

38. *To investigate a Rule for the Extraction of the Square Root of a compound Algebraical Quantity.*

Since the square of $a + b$ is $a^2 + 2ab + b^2$: in order to obtain the square root of $a^2 + 2ab + b^2$, we must consider by what process the quantity $a + b$ can be generally derived from it.

Now, in the first place, we observe that a, the first term of the root, is the square root of a^2 the first term of the square: and in addition to this, there still remains $2ab + b^2$ from which b is to be obtained: but $2ab + b^2$ is the same as $(2a + b)b$, and therefore b will be determined by dividing the first term of the remainder by *twice* the first term of the root, and to complete the operation, *twice* this first term

together with the second must be multiplied by the second, and after subtraction there is no remainder.

If the proposed quantity consist of more terms, it is evident that we have only to consider $a+b$ in the place of a, and thus by the same process another term of the root will be obtained, and so on: and hence we have the following general Rule.

Rule for the Extraction of the Square Root.

Arrange the terms in the order of the magnitudes of the indices of some one quantity: find the square root of the first term, and subtract its square from the proposed quantity: bring down the next two terms, and find the next term of the root by dividing this last quantity by twice the first, and affix it with its proper sign to the divisor: multiply this result by the said second term of the root: bring down to the remainder as many terms as may make the number equal to that in the next completed divisor; and thus continue the process till the root, or the requisite approximation to it, is obtained.

Ex. 1. Find the square root of $x^6 - 6x^3y^2 + 9y^4$.

Here,
$$\begin{array}{rl}
 & x^6 - 6x^3y^2 + 9y^4\ (x^3 - 3y^2 \\
 & x^6 \\ \hline
2x^3 - 3y^2) & -6x^3y^2 + 9y^4 \\
 & -6x^3y^2 + 9y^4 \\ \hline
\end{array}$$

whence the square root required is $x^3 - 3y^2$, as may easily be verified. Also, if the terms had been arranged in the reverse order, as, $9y^4 - 6x^3y^2 + x^6$, the root would have been found by a similar process to be $3y^2 - x^3$, which differs in its sign from the former, and is explained by the circumstance that the square root of a quantity is either positive or negativ as in Ex. (1) of article (37).

Ex. 2. Extract the square root of

$$4x^4 - 4x^3 - 3x^2 + 2x + 1.$$

Here,

$$
\begin{array}{r|l}
 & 4x^4 - 4x^3 - 3x^2 + 2x + 1 \,(2x^2 - x - 1 \\
 & 4x^4 \\ \hline
4x^2 - x) & -4x^3 - 3x^2 \\
 & -4x^3 + x^2 \\ \hline
4x^2 - 2x - 1) & -4x^2 + 2x + 1 \\
 & -4x^2 + 2x + 1 \\ \hline
\end{array}
$$

or, the root is $2x^2 - x - 1$: and to this, the remark of the last example may be applied.

Ex. 3. Find the square root of

$$16(a^4 + 1) - 24a(a^2 + 1) + 41a^2.$$

After arranging the terms according to the dimensions of a, we have

$$
\begin{array}{r|l}
 & 16a^4 - 24a^3 + 41a^2 - 24a + 16 \,(4a^2 - 3a + 4 \\
 & 16a^4 \\ \hline
8a^2 - 3a) & -24a^3 + 41a^2 \\
 & -24a^3 + 9a^2 \\ \hline
8a^2 - 6a + 4) & 32a^2 - 24a + 16 \\
 & 32a^2 - 24a + 16 \\ \hline
\end{array}
$$

The Extraction of the Square Roots of Numbers.

39. Inasmuch as with our present method of notation, numbers are not expressed in the same manner as the algebraical quantities which we have just been discussing, it is evident that the rule above laid down will not, without additional considerations, be sufficient for discovering their square roots. We shall presently see, however, that the same rule, assisted by what is called the *Method of Pointing*, will conduct us to the square roots of numerical magnitudes.

The Method of Pointing for the Square Root.

40. Since the square root of 1 is 1:
the square root of 100 is 10:
the square root of 10000 is 100:
the square root of 1000000 is 1000: &c.

we see immediately that the square root of a number of fewer than three figures must consist of only one figure: that of a number of more than two figures and fewer than five, of two figures: that of a number of more than four figures and fewer than seven, of three figures, and so on: whence it follows that if a dot or full point be placed over every alternate figure, beginning at the units' place, the number of such points will be the same as the number of figures in the square root.

The same rule may easily be extended to Decimals: thus,

since the square root of .01 is .1:
the square root of .0001 is .01:
the square root of .000001 is .001: &c.

we infer that the quantity proposed must first be made to have an *even* number of decimal places, and then that the pointing must be made from the place of units towards the right hand over every alternate figure, as before: and the number of such points will be the same as the number of decimal places in the square root.

Ex. 1. Extract the square root of 273529.

Arithmetical Form.

```
         · · ·
         273529 (523
         25
         ----
  102)   235
         204
         -----
 1043)   3129
         3129
         ----
```

Symbolical Form.

$$\begin{array}{rr}
 & 273529\ (500 + 20 + 3 \\
500^2 = & 250000 \\ \hline
2 \times 500 + 20 = 1020) & 23529 \\
 & 20400 \\ \hline
2 \times (500 + 20) + 3 = 1043) & 3129 \\
 & 3129 \\ \hline
\end{array}$$

Both these operations are performed by means of article (38): and a slight examination will shew that in reality they amount to the same thing, giving the same results, and differing only in consequence of the difference of the modes of expressing arithmetical and algebraical quantities.

Ex. 2. Required the square root of 19.0968.

Here,

$$\begin{array}{rl}
 & 19.0968\ (4.36 \\
 & 16 \\ \hline
83) & 309 \\
 & 249 \\ \hline
866) & 6068 \\
 & 5196 \\ \hline
 & 872 \\ \hline
\end{array}$$

In this case, the *approximate* square root is found by the common rule: and we remark, that as far as the operation has been continued, the last remainder is considerably larger than the last divisor, which can never be the case in arithmetical division: that this approximation to the root, however, is correct as far as it goes, may be made thus to appear.

If we suppose the root to be represented by the symbol a, the square will be represented by a^2: then, if we increase the numerical value of a by 1, the root becomes $a + 1$, and its square is $(a+1)^2 = a^2 + 2a + 1$: from which it is evident

that the root cannot admit of being augmented by 1, unless at the same time, the square is increased by $2a+1$, or by *twice* the original root $+1$: whence it follows that the approximate root is always correctly found as far as the operation is continued, whenever the remainder is not greater than *twice* the value of the said root: and thus the *Limit* of the remainder at any step is ascertained.

Examples for Practice.

(1) The square root of

$x^2y^4z^6$ is $\pm xy^2z^3$, and of $4x^{2m}y^{4n}z^{6p}$ is $\pm 2x^my^{2n}z^{3p}$.

(2) The cube root of

a^3b^6 is ab^2, and of $-8a^3b^6x^9$ is $-2ab^2x^3$.

(3) The fourth root of

$16a^4x^4$ is $\pm 2ax$, and of $81x^4y^{12}$ is $\pm 3xy^3$.

(4) The fifth root of $32a^5x^{10}y^{15}$ is $2ax^2y^3$, and the sixth root of $729x^6y^{18}z^{24}$ is $\pm 3xy^3z^4$.

(5) The square root of $x^4+2a^2x^2+a^4$ is x^2+a^2, and of $a^2x^2-2abxy^2+b^2y^4$ is $ax-by^2$.

(6) The square root of

$x^4-2x^3+3x^2-2x+1$ is x^2-x+1,

and of $4x^4-4x^3-3x^2+2x+1$ is $2x^2-x-1$.

(7) The square root of

$4a^2x^4-12a^3x^3+13a^4x^2-6a^5x+a^6$ is $2ax^2-3a^2x+a^3$, and of $9a^4x^2-12a^3x^3+10a^2x^4-4ax^5+x^6$ is $3a^2x-2ax^2+x^3$.

(8) The square root of

$4x^6-12x^5y+29x^4y^2-30x^3y^3+25x^2y^4$ is $2x^3-3x^2y+5xy^2$, and of

$4a^4-12a^3b+25a^2b^2-24ab^3+16b^4$ is $2a^2-3ab+4b^2$.

(9) The square root of

$9x^2y^4-12x^3y^3+34x^4y^2-20x^5y+25x^6$ is $3xy^c-2x^2y+5x^3$, and of

$4x^2y^4-12x^3y^3+17x^4y^2-12x^5y+4x^6$ is $2xy^2-3x^2y+2x^3$.

(10) The square root of

$$9 - 24x - 68x^2 + 112x^3 + 196x^4 \text{ is } 3 - 4x - 14x^2,$$

and of $9 - 6x + x^2 + 12x^3 - 4x^4 + 4x^6$ is $3 - x + 2x^3$.

(11) The square root of

$$x^2 - 2ax + a^2 + 2xy - 2ay + y^2 \text{ is } x - a + y,$$

and of $x^2 - 6xy + 2xz + 9y^2 - 6yz + z^2$ is $x - 3y + z$.

(12) The square root of

$$4x^4 - 12x^3 + 11x^2 - 3x + \tfrac{1}{4} \text{ is } 2x^2 - 3x + \tfrac{1}{2},$$

and of $49x^4 - 28x^3 - 17x^2 + 6x + \frac{9}{4}$ is $7x^2 - 2x - \frac{3}{2}$.

(13) The square root of

$$1 - 2x + 3x^2 - 4x^3 + 3x^4 - 2x^5 + x^6 \text{ is } 1 - x + x^2 - x^3,$$

and of

$$1 - 4x + 6x^2 - 6x^3 + 5x^4 - 2x^5 + x^6 \text{ is } 1 - 2x + x^2 - x^3.$$

(14) The square root of

$$9x^6 - 12x^5 + 10x^4 - 10x^3 + 5x^2 - 2x + 1 \text{ is } 3x^3 - 2x^2 + x - 1.$$

and of

$$x^8 - 4x^7 + 4x^6 - 4x^5 + 10x^4 - 4x^3 + 4x^2 - 4x + 1 \text{ is } x^4 - 2x^3 - 2x + 1.$$

(15) The square root of

$$36x^4 - 12(a - 2b)x^3 + (a^2 - 4ab + 4b^2)x^2 \text{ is } 6x^2 - (a - 2b)x,$$

and of

$$x^3(x - 2a) + a^2b(b - 2x) + (a^2 + 2ab)x^2 \text{ is } x^2 - a(x - b).$$

41. *To investigate a Rule for the Extraction of the cube root of a compound Algebraical Quantity.*

Since, $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, we must have the cube root of the latter quantity $= a + b$; and it remains to be determined in what manner it may be deduced from it.

Now, the first term a of the root is the cube root of a^3, the first term of the proposed quantity; hence, taking away

a^3, we have $3a^2b + 3ab^2 + b^3$ left to enable us to find b: but $3a^2b + 3ab^2 + b^3 = (3a^2 + 3ab + b^2)\,b$, and thence it is manifest that b will be obtained by dividing the first term of the remainder by *thrice* the square of a; and to complete the divisor, we must add to $3a^2$, *three* times the product of the two terms, $3ab$, and also the square of the last, b^2; thus, the second term being found, the repetition of a similar process will evidently lead to the root, whatever number of terms the expression may contain.

Ex. Find the cube root of $x^6 - 3x^5 + 5x^3 - 3x - 1$.

Here,

$$
\begin{array}{rl}
 & x^6 - 3x^5 + 5x^3 - 3x - 1 \;(\; x^2 - x - 1 \\
 & x^6 \\ \hline
3x^4 - 3x^3 + x^2\;) & -3x^5 + 5x^3 - 3x \\
 & -3x^5 + 3x^4 - x^3 \\ \hline
3x^4 - 6x^3 + 3x + 1\;) & -3x^4 + 6x^3 - 3x - 1 \\
 & -3x^4 + 6x^3 - 3x - 1 \\ \hline
\end{array}
$$

and thus the cube root $x^2 - x - 1$, is obtained in accordance with the principle of the article, the completed divisors being always formed in exact compliance with its directions.

42. The last article furnishes *a Rule*, which might, if necessary, be enunciated at length, in the same manner as that for the square root has been; and it is not difficult to perceive that a similar process may be applied to the extraction of the *fourth*, *fifth*, &c. roots of any compound quantity whatever.

Ex. To extract the cube root of 1860867.

Reasoning, analogous to that employed in article (40), will shew that if a point be placed over every third figure, beginning at the units' place, the number of points thus placed will be the number of digits in the cube root; and attention to article (41) will furnish the following operation:

$$
\begin{array}{lrl}
 & & a + b + c \\
 & \dot{1}86\dot{0}86\dot{7} & (100 + 20 + 3 \\
a^3 = & 1000000 & = \text{first subtrahend} \\
\hline
3a^2 = 30000) & 860867 & = \text{first remainder} \\
\hline
3a^2b = & 600000 & \\
3ab^2 = & 120000 & \\
b^3 = & 8000 & \\
\hline
 & 728000 & = \text{second subtrahend} \\
\hline
3(a+b)^2 \quad = 43200) & 132867 & = \text{second remainder} \\
\hline
3(a+b)^2c = & 129600 & \\
3(a+b)c^2 = & 3240 & \\
c^3 = & 27 & \\
\hline
 & 132867 & = \text{third subtrahend.} \\
\hline
\end{array}
$$

The cube root of 1860867 is therefore 123, and this process is the origin of the Rule given in article (161) of the Author's *Arithmetic*, to which the reader is now referred for additional Examples.

Examples for Practice.

(1) The cube root of $x^3 + 9x^2 + 27x + 27$ is $x + 3$, and of $1 - 6y + 12y^2 - 8y^3$ is $1 - 2y$.

(2) The cube root of $a^6 + 6a^5 - 40a^3 + 96a - 64$ is $a^2 + 2a - 4$,

and of

$$a^3 + b^3 + c^3 + 3(a^2b + a^2c + ab^2 + ac^2 + b^2c + bc^2) + 6abc$$

is $a + b + c$.

43. In the preceding article, it has been observed that the same principle may be applied to determine the *fourth*, *fifth*, &c. roots; but as the corresponding operations become exceedingly tedious, when the number of terms is large, we shall direct the Student's attention to the first *Appendix* of

the work, where he will find, among much other information, the extraction of any root of an *algebraical quantity* by a general method; and also, that of the cube root of a *number* consisting of several places of figures by a concise process; and it is scarcely necessary to remind him, that the *fourth* root of any quantity is the *square* root of the *square* root of that quantity: that the *sixth* root is the *cube* root of the *square* root; and so on.

Examples for Practice.

(1) The fourth root of $x^4 + 4x^3 + 6x^2 + 4x + 1$ is $x+1$, and of $a^4x^4 - 4a^3bx^3 + 6a^2b^2x^2 - 4ab^3x + b^4$ is $ax - b$.

(2) The sixth root of

$$x^6 - 12x^5 + 60x^4 - 160x^3 + 240x^2 - 192x + 64 \text{ is } x - 2,$$

and of $a^6 - 6a^5x + 15a^4x^2 - 20a^3x^3 + 15a^2x^4 - 6ax^5 + x^6$ is $a - x$.

VII. Miscellaneous Theorems.

44. Let x and y represent any two quantities whatever, whereof x is the greater; then if their sum be denoted by s, and their difference by d, we have

$$x + y = s, \text{ and } x - y = d;$$

$$\therefore\ s + d = (x + y) + (x - y) = 2x, \text{ or } x = \tfrac{1}{2}s + \tfrac{1}{2}d;$$

$$\text{and } s - d = (x + y) - (x - y) = 2y, \text{ or } y = \tfrac{1}{2}s - \tfrac{1}{2}d.$$

Hence, the greater of two quantities is always equal to half their sum increased by half their difference, and the less is equal to half their sum diminished by half their difference.

45. If each of the magnitudes a and b be divisible by c, then will $ma + nb$ and $ma - nb$, or $ma \pm nb$ be divisible by c, whatever whole numbers m and n may represent.

For, let $a = pc$ and $b = qc$, or a and b contain c, p and q times respectively; then we have

$$ma = mpc \text{ and } nb = nqc;$$

$$\text{whence, } ma \pm nb = mpc \pm nqc = (mp \pm nq)\,c,$$

or, $ma \pm nb$ contains c, $mp \pm nq$ times.

46. If the sum of two numbers be *given*, their product will be the greatest possible when they are *equal* to each other.

For, if $2a$ denote their sum, and $a+x$ be one of them, then the other will manifestly be

$$2a - (a + x) = a - x;$$

and their product will therefore be represented by

$$(a + x)(a - x) = a^2 - x^2,$$

which is evidently the greatest possible when $x = 0$, or when each of them $= a$, and their product $= a^2$.

47. Using the same symbols as in article (44), we have

$$(x + y) \times (x - y) = x^2 - y^2;$$

that is, the product of the sum and difference of any two quantities is always equal to the difference of the squares of the same quantities; and conversely.

This theorem is of great use, inasmuch as it will frequently enable us to represent an algebraical quantity by means of *factors*, when it is not so already, as in the following instances:

(1) $\quad a^2 - (b - c)^2 = (a + b - c)(a - b + c).$

(2) $\quad (a^2 + c^2)^2 - (a^2 - c^2)^2 - (a^2 + c^2 - b^2)^2$

$$= a^4 + 2a^2c^2 + c^4 - a^4 + 2a^2c^2 - c^4 - (a^2 + c^2 - b^2)^2$$

$$= 4a^2c^2 - (a^2 + c^2 - b^2)^2 = (2ac)^2 - (a^2 + c^2 - b^2)^2$$

$$= (2ac + a^2 + c^2 - b^2)(2ac - a^2 - c^2 + b^2)$$

$$= \{(a + c)^2 - b^2\}\{b^2 - (a - c)^2\}$$

$$= (a + b + c)(a + c - b)(b + a - c)(b - a + c).$$

(3) $$\frac{(a-b+c)\{(a+b)^2-c^2\}}{4b^2c^2-(a^2-b^2-c^2)^2}$$

$$=\frac{(a-b+c)(a+b+c)(a+b-c)}{(2bc)^2-(a^2-b^2-c^2)^2}$$

$$=\frac{(a-b+c)(a+b+c)(a+b-c)}{(2bc+a^2-b^2-c^2)(2bc-a^2+b^2+c^2)}$$

$$=\frac{(a-b+c)(a+b+c)(a+b-c)}{\{a^2-(b-c)^2\}\{(b+c)^2-a^2\}}$$

$$=\frac{(a-b+c)(a+b+c)(a+b-c)}{(a+b-c)(a-b+c)(b+c+a)(b+c-a)}$$

$=\dfrac{1}{b+c-a}$, by omitting the common factors.

48. Since $$(x+y)^2=x^2+2xy+y^2,$$
and $$(x-y)^2=x^2-2xy+y^2;$$
we observe that the square of the sum or difference of any two quantities, is equal to the sum of the squares of the quantities themselves, increased or diminished by twice their product.

Also, because $4\times x^2\times y^2=4x^2y^2=(\pm 2xy)^2$, we see that when a *trinomial* is a *complete square*, four times the product of the extreme terms is equal to the square of the mean.

THE RULE OF TRANSPOSITION.

49. If $a=b-c$; then, since when equals are added to, or subtracted from, equals, the results are equal, we shall have

$$a+c=b-c+c$$
$$=b:$$
$$a-b=b-c-b$$
$$=-c:$$
$$a-b+c=b-c-b+c$$
$$=0:$$

from which it appears that any quantity may be transposed from one side or member of an *Equality* to the other, by changing its algebraical sign from + to −, or from − to +.

On the same principle, if we have the *Inequalities* $a > b - c$, and $x < y + z$; then, by equal addition and subtraction, we find,

$$a + c > b - c + c$$
$$> b;$$
$$x - y < y + z - y$$
$$< z.$$

Also, similar conclusions will hold good, when both members are equally affected by the operations of Multiplication, Division, Involution or Evolution, provided the quantities under consideration be arithmetical.

Hence, because the *square* of every quantity whether positive or negative, has been proved to be positive, it follows that

$$(x - y)^2 \text{ or } x^2 - 2xy + y^2 > 0;$$
$$\therefore\ x^2 - 2xy + y^2 + 2xy > 0 + 2xy,$$
$$\text{or, } x^2 + y^2 > 2xy;$$

that is, the sum of the squares of any two *unequal* magnitudes, is always greater than twice their product.

CHAPTER III.

THE GREATEST COMMON MEASURES, AND LEAST COMMON MULTIPLES OF ALGEBRAICAL QUANTITIES.

I. THE GREATEST COMMON MEASURE.

50. DEF. *A Common Measure* of two or more quantities is a common *Factor*, which divides each of them without leaving a remainder; and the greatest common measure is the greatest factor by which they are so divisible.

Thus, of the quantities $2abd$ and $2dxy$, the factors 2, d and $2d$ are all common measures, the *greatest* or *highest* being $2d$: and $2d$ is said to *measure* $2abd$ and $2dxy$, by the *units* in ab and xy respectively.

Similarly, of $abcd$, $adey$ and $abdx$, the quantities a, d and ad are all common measures: but ad is the greatest in the sense intended in the definition, without reference to the numerical values that might be assigned to a and d.

51. COR. The greatest common factor of ad and bd is d, which is also the greatest common factor of acd and bd, or, of ad and bcd: that is, the greatest common measure of two quantities is the greatest common measure of either of them, and of the other when *multiplied* or *divided* by any quantity which is not a divisor of the first, and which contains no factor common to them both.

Also, if $-d$ be a common divisor of any number of quantities, then will $+d$ also divide them without remainders: and the greatest common measure is always supposed to appear in the form of a positive quantity.

52. When the quantities proposed are in the form of *Monomials*, the greatest common measure is readily discovered by inspection, as will appear in the following instances.

Ex. 1. The greatest common measure of $10ax$ and $15x^2$ is $5x$: because $10ax = 5x \times 2a$, and $15x^2 = 5x \times 3x$.

Ex. 2. The greatest common measure of

$$8a^2xy, \quad -12bxy^2, \text{ and } 20cx^2y \text{ is } 4xy:$$

since $8a^2xy = 4xy \times 2a^2$, $-12bxy^2 = 4xy \times (-3by)$, and $20cx^2y = 4xy \times 5cx$; the latter factors $2a^2$, $-3by$ and $5cx$ having no symbol in common.

Ex. 3. The greatest common measure of

$3(a+b)^3(c-x)^4$ and $5(a+b)^2(c-x)^5$ is $(a+b)^2(c-x)^4$, for a similar reason.

53. *To investigate a Rule for finding the greatest common measure of two compound Algebraical Quantities.*

Let a and b represent the two quantities where a is not of lower dimensions than b: and let b be contained p times in a with a remainder c: let c be contained q times in b with a remainder d, and let d be contained r times in c with no remainder, the operations being exhibited in the following form:

$$\begin{array}{llll}
b)\,a\,(p & & & \\
\;\;\,pb & & & \\
\hline
 & c)\,b\,(q & & \\
 & \;\;\,qc & & \\
\hline
 & & d)\,c\,(r & \\
 & & \;\;\,rd & \\
\hline
 & & \;\;\;0 &
\end{array}$$

then, d is the greatest common measure of a and b.

For, $c - rd = 0$, $\therefore$ we have $c = rd$, by transposition:

$$b - qc = d, \;\therefore\; b = d + qc = d + qrd = (1+qr)\,d:$$

$$a - pb = c, \;\therefore\; a = c + pb = rd + p(1+qr)\,d = (p + pqr + r)\,d:$$

from which it appears that d measures both a and b, and is therefore a common measure.

It is, moreover, the greatest common measure: for if not, let D be the greatest common measure, and let it be contained m and n times respectively in a and b, so that

$$a = mD, \text{ and } b = nD:$$

$$\therefore\ c = a - pb = mD - npD = (m - np)\,D:$$

$$\text{and } d = b - qc = nD - q(m - np)D = (n - mq + npq)D:$$

wherefore D measures d, or a greater quantity measures a less, which is absurd: and consequently no quantity but d is the greatest common measure.

In this investigation all the symbols have been treated as if they were arithmetical *integers*, and it is therefore a complete proof of the rule for finding the greatest common measure of two *numbers*, expressed in a *symbolical form*.

Moreover, when a and b are general symbols denoting compound algebraical quantities, it appears from the demonstration that if they contain any common factor whatever, it must also be found in each of the *partial* remainders c and d: and consequently whenever we arrive at a partial remainder containing a factor, which is *evidently* not a measure of each of the proposed quantities, that factor may be rejected, and the common measure will be contained in what is left.

This is manifest indeed from article (51), and forms a peculiar distinction between the arithmetical and algebraical processes: also, the latter may frequently be facilitated by the introduction of a new factor into a partial remainder, whenever the next quotient would otherwise be expressed in a fractional form, consistently with the observations made in the same article.

54. Cor. 1. Every common measure of two quantities, is a measure of their greatest common measure.

For, let δ be any common measure of a and b, so that

$$a = \mu\delta \text{ and } b = \nu\delta:$$

$$\text{then, } c = a - pb = \mu\delta - \nu p\delta = (\mu - \nu p)\delta:$$

$$\text{also, } d = b - qc = \nu\delta - q(\mu - \nu p)\delta = (\nu - \mu q + \nu pq)\delta:$$

and therefore δ is a measure of d.

55. Cor. 2. When the proposed quantities are *numbers*, it is obvious that each of the remainders c, d is less than that which immediately precedes it: and consequently in every case, this kind of division may be continued till the remainder becomes equal to zero: also, whenever the last divisor used happens to be 1, the numbers are said to be *prime* to each other.

56. The preceding investigation being put into words, furnishes the following general Rule.

Arrange the proposed quantities according to the *descending* dimensions of their characterizing symbol: divide the *higher* of them by the *lower*; and the preceding divisor by the last remainder: and continue the operation till there is no remainder, keeping in mind article (51): then will the last divisor be the greatest common measure.

Ex. 1. Find the greatest common measure of

$$x^2 + 2x + 1 \text{ and } x^3 + 2x^2 + 2x + 1.$$

$$\begin{array}{rl}
\text{Here, } x^2 + 2x + 1) & x^3 + 2x^2 + 2x + 1\,(x \\
 & \underline{x^3 + 2x^2 + x} \\
 & \qquad x + 1)\, x^2 + 2x + 1\,(x + 1 \\
 & \qquad\qquad\quad \underline{x^2 + x} \\
 & \qquad\qquad\qquad\quad x + 1 \\
 & \qquad\qquad\qquad\quad \underline{x + 1}
\end{array}$$

whence, $x + 1$ being the last divisor, is the greatest common measure of the quantities proposed.

Ex. 2. Required the greatest common measure of

$$x^4 + x^2y^2 + y^4 \text{ and } x^4 + 2x^3y + 3x^2y^2 + 2xy^3 + y^4.$$

Here,

$$\begin{array}{r|l}
x^4 + x^2y^2 + y^4 \,) & x^4 + 2x^3y + 3x^2y^2 + 2xy^3 + y^4 \,(\, 1 \\
& x^4 + x^2y^2 + y^4 \\ \hline
& 2x^3y + 2x^2y^2 + 2xy^3 \\ \hline
\end{array}$$

and this remainder being equivalent to $2xy(x^2 + xy + y^2)$, the factor $2xy$ may be rejected, because it is *evidently* not a common measure:

whence,

$$\begin{array}{r|l}
x^2 + xy + y^2 \,) & x^4 + x^2y^2 + y^4 \,(\, x^2 - xy + y^2 \\
& x^4 + x^3y + x^2y^2 \\ \hline
& -x^3y + y^4 \\
& -x^3y - x^2y^2 - xy^3 \\ \hline
& \qquad x^2y^2 + xy^3 + y^4 \\
& \qquad x^2y^2 + xy^3 + y^4 \\ \hline
\end{array}$$

and therefore $x^2 + xy + y^2$ is the greatest common measure.

Ex. 3. Find the highest common factor of the expressions

$$12x^3 + 4x^2 - 3x - 1 \text{ and } 8x^3 - 4x^2 - 2x + 1.$$

In this instance, the highest term of either of the quantities, when divided by that of the other, gives a result in the form of a fraction; and to avoid this, we multiply the latter by 3, and proceed as follows:

$$\begin{array}{r|l}
& 8x^3 - 4x^2 - 2x + 1 \\
& 3 \\ \hline
12x^3 + 4x^2 - 3x - 1 \,) & 24x^3 - 12x^2 - 6x + 3 \,(\, 2 \\
& 24x^3 + 8x^2 - 6x - 2 \\ \hline
& \qquad -20x^2 + 5 = -5(4x^2 - 1)
\end{array}$$

and of this remainder, the factor -5 being rejected, we have

$$4x^2-1\,)\,12x^3+4x^2-3x-1\,(\,3x+1$$
$$12x^3-3x$$
$$\overline{\quad 4x^2-1}$$
$$4x^2-1$$

so that $4x^2-1$ is the highest common factor.

Ex. 4. To find the greatest common measure of

$$2x^5-4x^4+8x^3-12x^2+6x \text{ and } 3x^5-3x^4-6x^3+9x^2-3x.$$

Here, we observe that x is found in every term, or

$$2x^5-4x^4+8x^3-12x^2+6x=2x\,(x^4-2x^3+4x^2-6x+3),$$
$$3x^5-3x^4-6x^3+9x^2-3x=3x\,(x^4-x^3-2x^2+3x-1):$$

so that x is a common measure which must finally be retained, but 2 and 3 are not, and may be immediately rejected:

now, $x^4-2x^3+4x^2-6x+3\,)\,x^4-x^3-2x^2+3x-1\,(\,1$

$$x^4-2x^3+4x^2-6x+3$$

$$x^3-6x^2+9x-4\,)\,x^4-2x^3+4x^2-6x+3\,(\,x+4$$
$$x^4-6x^3+9x^2-4x$$
$$\overline{4x^3-5x^2-2x+3}$$
$$4x^3-24x^2+36x-16$$

$$19x^2-38x+19=19\,(x^2-2x+1)$$

whence, $x^2-2x+1\,)\,x^3-6x^2+9x-4\,(\,x-4$

$$x^3-2x^2+x$$
$$\overline{-4x^2+8x-4}$$
$$-4x^2+8x-4$$

and, therefore, $x^2 - 2x + 1$ being the greatest common measure of these latter factors, the greatest common measure of the quantities proposed will be

$$x(x^2 - 2x + 1), \text{ or } x^3 - 2x^2 + x.$$

57. Whenever the quantities proposed involve more *literal* symbols than one, and the terms are not homogeneous, the application of the Rule laid down in the last article will generally be found to be attended with some difficulty, in consequence of the existence of compound factors which are not common measures, in the partial remainders. In such cases, it will be best to have recourse to the resolution of the quantities into their constituent factors, by which the common measures will be immediately made apparent.

Ex. 1. Find the highest common factor of

$$9x^2 - 3xy - 6x + 2y \text{ and } 6x^4 - 4x^3 - 3xy^2 + 2y^2.$$

Here, $9x^2 - 3xy - 6x + 2y = (9x^2 - 3xy) - (6x - 2y)$

$$= 3x(3x - y) - 2(3x - y) = (3x - 2)(3x - y):$$

$$6x^4 - 4x^3 - 3xy^2 + 2y^2 = (6x^4 - 4x^3) - (3xy^2 - 2y^2)$$

$$= 2x^3(3x - 2) - y^2(3x - 2) = (3x - 2)(2x^3 - y^2):$$

so that the factor required is $3x-2$, inasmuch as $3x-y$ and $2x^3 - y^2$ have evidently no common measure.

Ex. 2. Of the two quantities

$$3bcz+5mxz+30mx+18bc \text{ and } 4adz-7vrz+24ad-42vr,$$

we observe that

$$\text{the former} = 3bc(z + 6) + 5mx(z + 6)$$

$$= (3bc + 5mx)(z + 6):$$

$$\text{the latter} = 4ad(z + 6) - 7vr(z + 6)$$

$$= (4ad - 7vr)(z + 6):$$

and therefore the greatest common factor is $z + 6$.

Ex. 3. Required the greatest common measure of

$$a^3 + a^2b - ab^2 - b^3 \text{ and } a^3 - a^2b - ab^2 + b^3.$$

$$\text{Here, } a^3 + a^2b - ab^2 - b^3 = (a^3 + a^2b) - (ab^2 + b^3)$$
$$= a^2(a + b) - b^2(a + b) = (a^2 - b^2)(a + b):$$
$$\text{and } a^3 - a^2b - ab^2 + b^3 = (a^3 - a^2b) - (ab^2 - b^3)$$
$$= a^2(a - b) - b^2(a - b) = (a^2 - b^2)(a - b):$$

and since $a + b$ and $a - b$ have no factor in common, the greatest common measure is $a^2 - b^2$.

This example shews that the principle of the article is general, and may be extended to all cases where it is easy of application.

58. *To find the greatest common measure of three or more quantities.*

Let a, b, c be any three proposed quantities, and suppose the greatest common measure of a and b to be d: then the greatest common measure of d and c will be the greatest common measure of a, b and c.

For, since d is the greatest common measure of a and b, every measure of d is a common measure of a and b by article (54): therefore every common measure of d and c is a common measure of a, b, and c: and consequently the greatest common measure of d and c, is the greatest common measure of a, b, and c.

In the same manner, whatever be the number of quantities, their greatest common factor will be determined by a continuation of this process.

Ex. Find the highest common factor of
$a^3 + a^2b - ab^2 - b^3$, $a^3 - 2a^2b - ab^2 + 2b^3$ and $a^3 - 3ab^2 + 2b^3$.

In the first place, to find the highest common factor of the two quantities $a^3 + a^2b - ab^2 - b^3$ and $a^3 - 2a^2b - ab^2 + 2b^3$, we have

$$
\begin{array}{r|l}
a^3 + a^2b - ab^2 - b^3 & a^3 - 2a^2b - ab^2 + 2b^3 \;(1 \\
 & a^3 + a^2b \;\; - ab^2 - b^3 \\ \hline
 & -3a^2b + 3b^3 = -3b(a^2 - b^2)
\end{array}
$$

$$
\begin{array}{l}
a^2 - b^2 \big) \, a^3 + a^2b - ab^2 - b^3 \big(a + b \\
\qquad\quad \underline{a^3 - ab^2} \\
\qquad\qquad\quad a^2b - b^3 \\
\qquad\qquad\quad \underline{a^2b - b^3}
\end{array}
$$

so that $a^2 - b^2$ is the highest common factor of the first two quantities, and it remains to find the same of this and the third: thus,

$$
\begin{array}{l}
a^2 - b^2 \big) \, a^3 - 3ab^2 + 2b^3 \big(a \\
\qquad\quad \underline{a^3 - ab^2} \\
\qquad\qquad - 2ab^2 + 2b^3 = - 2b^2 (a - b) \\
a - b \big) \, a^2 - b^2 \big(a + b \\
\qquad \underline{a^2 - ab} \\
\qquad\qquad ab - b^2 \\
\qquad\qquad \underline{ab - b^2}
\end{array}
$$

wherefore $a - b$ is the highest common factor required.

II. THE LEAST COMMON MULTIPLE.

59. Def. A *Common Multiple* of two or more quantities, is another quantity which is capable of being divided by each of them without leaving any remainder: and the least or lowest common multiple is the least or lowest quantity which each of them can divide exactly.

Thus, $2abc$ is a common multiple of ab and bc, and abc is their least common multiple; so also, $3abx$ is the least common multiple of $3a$, $3bx$ and abx.

60. Cor. Hence, the least common multiple of two or more quantities which have no common factor except 1, is their product.

61. The least common multiples of monomials, and of compound quantities involving common factors which are *explicitly* exhibited, may generally be found by inspection, as in the following instances.

Ex. 1. The least common multiple of a^2bc and $2ab^2d$, is $2a^2b^2cd$: for $a^2bc = ab(ac)$ and $2ab^2d = ab(2bd)$.

Ex. 2. The lowest common multiple of axy, a^2y and $ax+by$, is $a^3x^2y + a^2bxy^2$.

Ex. 3. The least common multiple of

$$a^2(x+y),\ ab(x-y) \text{ and } x^2-y^2, \text{ is } a^2b(x^2-y^2),$$
$$\text{because } x^2-y^2 = (x+y)(x-y).$$

62. *To investigate a Rule for finding the least common multiple of two Algebraical Quantities.*

Let a and b represent the two quantities, and d their greatest common measure such that $a = pd$ and $b = qd$; then, since p and q have no common measure except 1, their least common multiple will be pq, by article (60); wherefore pqd will manifestly be the least common multiple of pd and qd, or of a and b; that is, denoting the least common multiple by m, we shall have

$$m = pqd = \frac{pd \times qd}{d} = \frac{ab}{d};$$

or, the least common multiple $= \dfrac{\text{the product}}{\text{the greatest common measure}}$.

63. Cor. Every common multiple of two quantities is a multiple of, or exactly divisible by, the least common multiple.

For, let μ be any common multiple of a and b, and if possible, let m be contained r times in μ with a remainder s, so that

$$\frac{\mu}{m} = r + \frac{s}{m}, \text{ or } \mu = rm + s;\ \therefore\ s = \mu - rm;$$

wherefore, since a and b measure μ and m, they will also measure s, which is less than m; that is, m is not the least common multiple of a and b, contrary to the supposition: hence every other common multiple is a multiple of the least common multiple.

Ex. 1. Required the least common multiple of $a^3 + a^2b$ and $a^2 - b^2$.

Here, $a^3 + a^2b = a^2(a+b)$, and $a^2 - b^2 = (a-b)(a+b)$;

$$\therefore \text{ the least common multiple} = \frac{a^2(a+b) \times (a-b)(a+b)}{a+b}$$

$$= a^2 \times (a-b)(a+b) = a^2(a^2 - b^2) = a^4 - a^2b^2,$$

because their greatest common measure is $a + b$.

Ex. 2. Find the least common multiple of $x^3 + x^2 + x + 1$ and $x^3 - x^2 + x - 1$.

By the ordinary process, the greatest common measure is $x^2 + 1$; whence the least common multiple will be

$$\frac{(x^3 + x^2 + x + 1)(x^3 - x^2 + x - 1)}{x^2 + 1}$$

$$= (x^3 + x^2 + x + 1)\left(\frac{x^3 - x^2 + x - 1}{x^2 + 1}\right)$$

$$= (x^3 + x^2 + x + 1)(x - 1) = x^4 - 1.$$

64. *To find the least common multiple of three or more quantities.*

Let a, b, c be the proposed quantities, and let m be the least common multiple of a and b; then the least common multiple of m and c, will be the least common multiple of a, b and c.

For, since m is the least common multiple of a and b, every multiple of m is a common multiple of a and b; and every common multiple of m and c is a common multiple of a, b and c; whence it follows that the least common multiple of m and c is the least common multiple of a, b and c.

The same kind of reasoning is manifestly applicable, whatever be the number of quantities proposed.

Ex. Required the lowest common multiple of $a^2 + ab$, $a^4 + a^2b^2$ and $a^4 - b^4$.

Here, $a^2 + ab = a(a + b)$, and $a^4 + a^2b^2 = a^2(a^2 + b^2)$; therefore the lowest common multiple of the first two, is

$$(a + b) \times a^2(a^2 + b^2) = a^2(a + b)(a^2 + b^2);$$

and since $a^4 - b^4 = (a^2 - b^2)(a^2 + b^2) = (a + b)(a - b)(a^2 + b^2)$, the lowest common multiple required will evidently be

$$a^2(a^2 - b^2)(a^2 + b^2) = a^2(a^4 - b^4) = a^6 - a^2b^4.$$

65. The last article is of very great practical use in the treatment of both arithmetical and algebraical fractions, in the latter of which it will generally enable us to diminish both the number and the orders of the symbols employed.

Ex. Transform $\frac{1}{12}$, $\frac{1}{16}$, $\frac{1}{21}$ and $\frac{1}{60}$, so as to have the least common denominator.

Here, $\frac{12 \times 16}{4} = 48$, $\frac{48 \times 21}{3} = 336$, and $\frac{336 \times 60}{12} = 1680$:

$$\text{whence, } \left.\begin{aligned} \frac{1}{12} &= \frac{1 \times 140}{12 \times 140} = \frac{140}{1680} \\ \frac{1}{16} &= \frac{1 \times 105}{16 \times 105} = \frac{105}{1680} \\ \frac{1}{21} &= \frac{1 \times 80}{21 \times 80} = \frac{80}{1680} \\ \frac{1}{60} &= \frac{1 \times 28}{60 \times 28} = \frac{28}{1680} \end{aligned}\right\} \text{the new equivalent fractions having the least common denominator.}$$

CHAPTER IV.

ALGEBRAICAL FRACTIONS.

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66. In Arithmetical Algebra, the nature and character of fractions is precisely the same as in Arithmetic; and of course, the operations upon them will be conducted in the same manner. The most important of these operations have been symbolically expressed in the subdivisions of article (10), and in performing them, the chief thing to be attended to, is their reduction so as to involve symbols the fewest in number and of the lowest order. We will give instances of these reduced processes, in the following articles.

67. *To represent any quantity in the form of a fraction.*

Let a be the quantity proposed, then we have

$$a = \frac{a}{1} = \frac{2a}{2} = \frac{3a}{3} = \&c. = \frac{ad}{d} = \frac{-ad}{-d}:$$

and from this it appears that the value of a fraction is not altered by changing the signs of the numerator and denominator, which is in fact the same thing as multiplying each of them by -1: and the converse.

68. *To represent a mixed quantity in the form of a fraction.*

This is the result expressed in (3) of article (10),

$$\text{where } a + \frac{b}{c} = \frac{ac+b}{c}, \text{ and } a - \frac{b}{c} = \frac{ac-b}{c}.$$

Ex. 1. Reduce $a + \dfrac{(a-x)^2}{4x}$ to a fractional form.

$$\text{Here, } a + \frac{(a-x)^2}{4x} = \frac{4ax + a^2 - 2ax + x^2}{4x}$$

$$= \frac{a^2 + 2ax + x^2}{4x} = \frac{(a+x)^2}{4x}.$$

Ex. 2. Find the fraction equivalent to $a+b-\dfrac{2ab+b^2}{a+b}$.

$$\text{Here, } a + b - \frac{2ab+b^2}{a+b} = \frac{(a+b)^2 - (2ab+b^2)}{a+b}$$

$$= \frac{a^2 + 2ab + b^2 - 2ab - b^2}{a+b} = \frac{a^2}{a+b}.$$

69. *To represent, when possible, a fraction in the form of a mixed quantity.*

From (4) of article (10), we have $\dfrac{ac+b}{c} = a + \dfrac{b}{c}$, and $\dfrac{ac-b}{c} = a - \dfrac{b}{c}$, which indicate the process to be adopted.

Ex. 1. The fraction $\dfrac{a^2 + 4ab + 4b^2}{a} = a + 4b + \dfrac{4b^2}{a}$.

Ex. 2. The fraction $\dfrac{a^3 - b^3 + x^3}{a+x}$ is equivalent to the mixed quantity $a^2 - ax + x^2 - \dfrac{b^3}{a+x}$, by actual division.

70. *To reduce a fraction to its lowest terms, or simplest form.*

The process required here is obvious, for a fraction will always be reduced to its simplest form by dividing the numerator and denominator by their greatest common factor.

Ex. 1. The fraction $\dfrac{a^2xy^2}{4a^2x^2y} = \dfrac{y}{4x}$, by dividing the numerator and denominator by their greatest common measure a^2xy.

Ex. 2. In the fraction $\frac{6a^2 + 5ax - 6x^2}{6a^2 + 13ax + 6x^2}$, the greatest common factor is found to be $2a + 3x$: and the numerator and denominator being divided by it, we have the fraction expressed in its simplest terms $= \frac{3a - 2x}{3a + 2x}$.

71. *To reduce fractions to others having a common denominator.*

Let $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{e}{f}$ be the fractions proposed:

$$\text{then, } \frac{a}{b} = \frac{a \times d \times f}{b \times d \times f} = \frac{adf}{bdf}:$$

$$\frac{c}{d} = \frac{c \times b \times f}{d \times b \times f} = \frac{cbf}{bdf}:$$

$$\frac{e}{f} = \frac{e \times b \times d}{f \times b \times d} = \frac{ebd}{bdf}:$$

and the equivalent fractions are $\frac{adf}{bdf}$, $\frac{cbf}{bdf}$ and $\frac{ebd}{bdf}$, having the common denominator bdf.

Ex. 1. Reduce $\frac{a}{2b}$, $\frac{3b}{a}$ and $\frac{5xy}{c}$, so as to have a common denominator.

$$\text{Here, } \frac{a}{2b} = \frac{a \times a \times c}{2b \times a \times c} = \frac{a^2 c}{2abc}:$$

$$\frac{3b}{a} = \frac{3b \times 2b \times c}{a \times 2b \times c} = \frac{6b^2 c}{2abc}:$$

$$\frac{5xy}{c} = \frac{5xy \times 2b \times a}{c \times 2b \times a} = \frac{10abxy}{2abc}:$$

the resulting fractions having the common denominator $2abc$.

Ex. 2. Express $\frac{a+x}{a-x}$ and $\frac{a-x}{a+x}$ as fractions having a common denominator.

Here, $\dfrac{a+x}{a-x} = \dfrac{(a+x)\times(a+x)}{(a-x)\times(a+x)} = \dfrac{a^2+2ax+x^2}{a^2-x^2}$:

$$\frac{a-x}{a+x} = \frac{(a-x)\times(a-x)}{(a+x)\times(a-x)} = \frac{a^2-2ax+x^2}{a^2-x^2}:$$

and these fractions are of the same values as those proposed, but expressed in such forms as to have the common denominator a^2-x^2.

72. COR. If the denominators of the proposed fractions have a common measure, the fractions may be transformed into others having a lower common denominator than what is determined by the foregoing process: thus, if the fractions be $\dfrac{a}{bd}$ and $\dfrac{c}{de}$, we have immediately,

$$\frac{a}{bd} = \frac{ae}{bde}, \text{ and } \frac{c}{de} = \frac{bc}{bde}:$$

from which it appears that the least common denominator is the least common multiple of the proposed denominators: and the numerators are obtained by multiplying the original numerators by the quotients arising from the division of the least common multiple by the corresponding denominators.

Ex. Reduce $\dfrac{x^2}{a^2+ax}$, $\dfrac{a^2}{ax-x^2}$ and $\dfrac{ax}{a^2-x^2}$ to equivalent fractions, having the lowest common denominator.

Here, we find the lowest common multiple of

$a(a+x)$, $x(a-x)$ and $(a-x)(a+x)$, to be $ax(a^2-x^2)$: whence we have

$$\frac{x^2}{a(a+x)} = \frac{x^2\times x(a-x)}{ax(a^2-x^2)} = \frac{ax^3-x^4}{ax(a^2-x^2)}:$$

$$\frac{a^2}{x(a-x)} = \frac{a^2\times a(a+x)}{ax(a^2-x^2)} = \frac{a^4+a^3x}{ax(a^2-x^2)}:$$

$$\frac{ax}{a^2-x^2} = \frac{ax\times ax}{ax(a^2-x^2)} = \frac{a^2x^2}{ax(a^2-x^2)}.$$

73. *To find the sum and difference of two fractions.*

Let $\frac{a}{b}$ and $\frac{c}{d}$ be the proposed fractions, and assume $\frac{a}{b}$ and $\frac{c}{d}$ to be represented by the simple symbols x and y,

$$\text{or, } \frac{a}{b} = x \text{ and } \frac{c}{d} = y:$$

then by equal multiplication, we shall have

$$a = bx \text{ and } c = dy:$$

$$\therefore\ ad = bdx \text{ and } bc = bdy:$$

$$\text{whence, } ad + bc = bdx + bdy = bd(x + y),$$

$$\text{that is, } x + y, \text{ or } \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}:$$

$$\text{and, } ad - bc = bdx - bdy = bd(x - y),$$

$$\text{that is, } x - y, \text{ or } \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}.$$

These results are the same as (6) and (7) of article (10): and if they were put into words, amount to the same thing as the Arithmetical rules for these operations.

It is also evident that a similar process may be applied whatever be the number of fractions considered.

Ex. 1. Find the sum of $\frac{a}{a+x}$ and $\frac{x}{a-x}$.

$$\text{Here, } \frac{a}{a+x} + \frac{x}{a-x} = \frac{a(a-x)}{(a+x)(a-x)} + \frac{x(a+x)}{(a+x)(a-x)}$$

$$= \frac{a^2 - ax + ax + x^2}{a^2 - x^2} = \frac{a^2 + x^2}{a^2 - x^2}.$$

Ex. 2. Simplify as much as possible, the expression

$$\frac{1}{4a^3(a+x)} + \frac{1}{4a^3(a-x)} + \frac{1}{2a^2(a^2+x^2)}.$$

Here, $$\frac{1}{4a^3(a+x)} + \frac{1}{4a^3(a-x)} + \frac{1}{2a^2(a^2+x^2)}$$

$$= \frac{1}{4a^3}\left\{\frac{1}{a+x} + \frac{1}{a-x}\right\} + \frac{1}{2a^2(a^2+x^2)}$$

$$= \frac{1}{4a^3}\left\{\frac{a-x+a+x}{a^2-x^2}\right\} + \frac{1}{2a^2(a^2+x^2)}$$

$$= \frac{1}{4a^3}\left\{\frac{2a}{a^2-x^2}\right\} + \frac{1}{2a^2(a^2+x^2)}$$

$$= \frac{1}{2a^2}\left\{\frac{1}{a^2-x^2}\right\} + \frac{1}{2a^2(a^2+x^2)}$$

$$= \frac{1}{2a^2}\left\{\frac{1}{a^2-x^2} + \frac{1}{a^2+x^2}\right\}$$

$$= \frac{1}{2a^2}\left\{\frac{a^2+x^2+a^2-x^2}{a^4-x^4}\right\}$$

$$= \frac{1}{2a^2}\left\{\frac{2a^2}{a^4-x^4}\right\} = \frac{1}{a^4-x^4}.$$

Ex. 3. Subtract $\frac{x-y}{x+y}$ from $\frac{x+y}{x-y}$.

Here, $$\frac{x+y}{x-y} - \frac{x-y}{x+y} = \frac{(x+y)^2-(x-y)^2}{x^2-y^2}$$

$$= \frac{(x^2+2xy+y^2)-(x^2-2xy+y^2)}{x^2-y^2} = \frac{4xy}{x^2-y^2}.$$

Ex. 4. Express in its simplest form, the quantity

$$\frac{x+y}{y} - \frac{2x}{x+y} + \frac{x^3-x^2y}{y^3-x^2y}.$$

Here, $$\frac{x+y}{y} - \frac{2x}{x+y} + \frac{x^3 - x^2y}{y^3 - x^2y}$$

$$= \frac{x+y}{y} - \frac{2x}{x+y} + \frac{x^2(x-y)}{y(y^2-x^2)}$$

$$= \frac{x+y}{y} - \frac{2x}{x+y} - \frac{x^2(x-y)}{y(x^2-y^2)}$$

$$= \frac{x+y}{y} - \frac{2x}{x+y} - \frac{x^2}{y(x+y)}$$

$$= \frac{(x+y)^2}{y(x+y)} - \frac{2xy}{y(x+y)} - \frac{x^2}{y(x+y)}$$

$$= \frac{x^2 + 2xy + y^2 - 2xy - x^2}{y(x+y)} = \frac{y^2}{y(x+y)} = \frac{y}{x+y}.$$

74. *To find the product and quotient of two fractions.*

Let $\frac{a}{b}$ and $\frac{c}{d}$ represent the two fractions, and as before assume $\frac{a}{b} = x$ and $\frac{c}{d} = y$:

$$\text{then, } a = bx \text{ and } c = dy:$$

$$\text{whence, } a \times c = bx \times dy, \text{ or } ac = bdxy;$$

$$\therefore\ xy, \text{ or } \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} = \frac{a \times c}{b \times d},$$

which gives the arithmetical rule for the multiplication of fractions, indicated in (8) of article (10):

$$\text{also, since } a = bx \text{ and } c = dy, \text{ we have}$$

$$ad = bdx \text{ and } bc = bdy:$$

$$\therefore \text{ by equal division, } \frac{bdx}{bdy} = \frac{ad}{bc}:$$

$$\text{that is, } \frac{x}{y}, \text{ or } \frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc} = \frac{a}{b} \times \frac{d}{c},$$

which furnishes the rule of common Arithmetic, to invert the divisor and proceed as in multiplication, as expressed in (9) of article (10).

Ex. 1. Find the product of $\frac{ax}{(a-x)^2}$ and $\frac{a^2-x^2}{ab}$.

Here, $\frac{ax}{(a-x)^2} \times \frac{a^2-x^2}{ab} = \frac{ax \times (a^2-x^2)}{(a-x)^2 \times ab}$.

$$= \frac{ax \times (a+x)(a-x)}{(a-x)(a-x) \times ab} = \frac{x(a+x)}{b(a-x)} = \frac{ax+x^2}{ab-bx}.$$

Ex. 2. Multiply $\frac{a}{x} + \frac{b}{y}$ by $\frac{x^2}{a^2} + \frac{y^2}{b^2}$.

Here,

$$\begin{array}{l} \frac{a}{x} + \frac{b}{y} \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} \\ \hline \frac{x}{a} + \frac{bx^2}{a^2y} \\ \qquad\quad \frac{ay^2}{b^2x} + \frac{y}{b} \\ \hline \frac{x}{a} + \frac{bx^2}{a^2y} + \frac{ay^2}{b^2x} + \frac{y}{b} \\ \hline \end{array}$$

which is the same result as would have been obtained from multiplying together $\frac{ay+bx}{xy}$ and $\frac{b^2x^2+a^2y^2}{a^2b^2}$, the values of the multiplier and multiplicand expressed in different forms, and then rejecting from the terms of the numerator and denominator, such factors as are common.

vide $\frac{2ax-x^2}{c^3-x^3}$ by $\frac{2a-x}{(c-x)^2}$.

Here, $\dfrac{2ax - x^2}{c^3 - x^3} \div \dfrac{2a - x}{(c-x)^2} = \dfrac{2ax - x^2}{c^3 - x^3} \times \dfrac{(c-x)^2}{2a - x}$

$$= \frac{(2a - x)x}{(c-x)(c^2 + cx + x^2)} \times \frac{(c-x)(c-x)}{2a - x}$$

$$= \frac{(2a-x)x \times (c-x)(c-x)}{(c-x)(c^2+cx+x^2) \times (2a-x)} = \frac{x(c-x)}{c^2+cx+x^2} = \frac{cx - x^2}{c^2+cx+x^2},$$

by rejecting from the numerator and denominator, those factors which are common to both.

Ex. 4. Divide $\dfrac{8ab}{9x^2} + 2 + \dfrac{9x^2}{8ab}$ by $\dfrac{4a}{3x} + \dfrac{3x}{2b}$.

Here, $\left.\dfrac{4a}{3x} + \dfrac{3x}{2b}\right) \dfrac{8ab}{9x^2} + 2 + \dfrac{9x^2}{8ab} \left(\dfrac{2b}{3x} + \dfrac{3x}{4a}\right.$

$$\frac{8ab}{9x^2} + 1$$

$$1 + \frac{9x^2}{8ab}$$

$$1 + \frac{9x^2}{8ab}$$

where the quotient is the same as would be obtained from dividing $\dfrac{64a^2b^2 + 144abx^2 + 81x^4}{72abx^2}$ by $\dfrac{8ab + 9x^2}{6bx}$, and simplifying the terms of the result by reduction.

75. *To find the powers and roots of a fraction.*

Let $\dfrac{a}{b}$ be the proposed fraction, then will the m^{th} power of $\dfrac{a}{b}$ be $\dfrac{a^m}{b^m}$, in accordance with article (12): and this gives the common practical rule.

Again, by reversing the process implied in the last operation, we have the m^{th} root of $\dfrac{a^m}{b^m}$ equal to $\dfrac{a}{b}$, from which a general rule is immediately derived.

Ex. 1. The square of $\dfrac{x+2y}{x-y}$ will be

$$\frac{(x+2y)(x+2y)}{(x-y)(x-y)} = \frac{x^2+4xy+4y^2}{x^2-2xy+y^2}:$$

the cube of $\dfrac{x+2y}{x-y} = \dfrac{x^3+6x^2y+12xy^2+8y^3}{x^3-3x^2y+3xy^2-y^3}$.

Ex. 2. The square root of $\dfrac{a^4-2a^2x^2+x^4}{a^2+2ax+x^2}$

$$= \frac{\text{the square root of } a^4-2a^2x^2+x^4}{\text{the square root of } a^2+2ax+x^2} = \frac{a^2-x^2}{a+x} = a-x.$$

Ex. 3. Required the square root of $\dfrac{a^2}{b^2} - \dfrac{4a}{3c} + \dfrac{4b^2}{9c^2}$.

Here, $\quad \dfrac{a^2}{b^2} - \dfrac{4a}{3c} + \dfrac{4b^2}{9c^2} \left(\dfrac{a}{b} - \dfrac{2b}{3c} \right.$

$$\frac{a^2}{b^2}$$

$$\left. \frac{2a}{b} - \frac{2b}{3c} \right) - \frac{4a}{3c} + \frac{4b^2}{9c^2}$$

$$- \frac{4a}{3c} + \frac{4b^2}{9c^2}$$

In this example, we might have reduced the terms of the quantity proposed to a common denominator, and then have extracted the root of the numerator and denominator separately.

MISCELLANEOUS THEOREMS.

76. If $\frac{a}{b} = \frac{c}{d}$, it is required to prove that $\frac{a+b}{a-b} = \frac{c+d}{c-d}$.

Since $\frac{a}{b} = \frac{c}{d}$, we have $\frac{a}{b} \pm 1 = \frac{c}{d} \pm 1$:

that is, $\frac{a+b}{b} = \frac{c+d}{d}$, and $\frac{a-b}{b} = \frac{c-d}{d}$:

whence, $\frac{a+b}{b} \div \frac{a-b}{b} = \frac{c+d}{d} \div \frac{c-d}{d}$:

$\therefore \frac{a+b}{b} \times \frac{b}{a-b} = \frac{c+d}{d} \times \frac{d}{c-d}$, or $\frac{a+b}{a-b} = \frac{c+d}{c-d}$.

Hence, also conversely, if $\frac{a+b}{a-b} = \frac{c+d}{c-d}$: then will $\frac{a}{b} = \frac{c}{d}$.

77. If $\frac{a}{b} = \frac{c}{d}$, then will $\frac{ma \pm nb}{pa \pm qb} = \frac{mc \pm nd}{pc \pm qd}$.

For, since $\frac{a}{b} = \frac{c}{d}$, we have $\frac{ma}{nb} = \frac{mc}{nd}$:

$\therefore \frac{ma}{nb} \pm 1 = \frac{mc}{nd} \pm 1$, or $\frac{ma \pm nb}{nb} = \frac{mc \pm nd}{nd}$:

and by a similar process, $\frac{pa \pm qb}{qb} = \frac{pc \pm qd}{qd}$:

whence, by equal division, we obtain

$$\frac{ma \pm nb}{nb} \div \frac{pa \pm qb}{qb} = \frac{mc \pm nd}{nd} \div \frac{pc \pm qd}{qd},$$

or, $\frac{ma \pm nb}{pa \pm qb} = \frac{mc \pm nd}{pc \pm qd}$, by rejecting the common factors.

78. To find whether $\frac{a+x}{b+x}$ is greater or less than $\frac{a}{b}$.

Here, we have $\frac{a+x}{b+x} = \frac{b(a+x)}{b(b+x)} = \frac{ab+bx}{b(b+x)}$,

and $\frac{a}{b} = \frac{a(b+x)}{b(b+x)} = \frac{ab+ax}{b(b+x)}$:

by reducing the fractions to a common denominator:

$$\text{whence, } \frac{a+x}{b+x} \text{ is } > \text{ or } < \frac{a}{b},$$

$$\text{according as } \frac{ab+bx}{b(b+x)} \text{ is } > \text{ or } < \frac{ab+ax}{b(b+x)},$$

according as $ab + bx$ is $>$ or $<$ $ab + ax$,

according as bx is $>$ or $< ax$, as b is $>$ or $<$ a.

Similarly, $\frac{a-x}{b-x} = \frac{b(a-x)}{b(b-x)} = \frac{ab-bx}{b(b-x)}$,

and $\frac{a}{b} = \frac{a(b-x)}{b(b-x)} = \frac{ab-ax}{b(b-x)}$:

$$\text{whence it follows that } \frac{a-x}{b-x} \text{ is } > \text{ or } < \frac{a}{b},$$

according as $ab - bx$ is $>$ or $<$ $ab - ax$,

according as $ab - bx + bx + ax$ is $>$ or $<$ $ab - ax + bx + ax$,

according as $ab + ax$ is $>$ or $<$ $ab + bx$,

according as ax is $>$ or $<$ bx, as a is $>$ or $<$ b.

In this last case, we have supposed x to be less than a or b, in order that the values of the fractions may be arithmetical; and it is observed that the less the value of a number is when taken positively, the greater it is when taken negatively: and *vice versâ*.

79. Cor. Hence also, if $\frac{a}{b}$ > or < $\frac{c}{d}$, then is $\frac{ma}{b}$ > or < $\frac{mc}{d}$, and $\left(\frac{a}{b}\right)^m$ > or < $\left(\frac{c}{d}\right)^m$, according as m is positive or negative.

80. If we have an equation, as $\frac{a}{b} = \frac{c}{d}$, and both its members be multiplied by bd, we shall have

$$\frac{a}{b} \times bd = \frac{c}{d} \times bd, \text{ or } ad = bc:$$

from which we infer that an equation may be *cleared of Fractions*, by multiplying all its terms by the product of their denominators, or by their least common multiple.

Thus, from $\frac{ax}{b+x} - \frac{bx}{a+x} = c$, we have

$$ax(a+x) - bx(b+x) = c(b+x)(a+x):$$

$$\text{or, } a^2x + ax^2 - b^2x - bx^2 = abc + (a+b)cx + cx^2:$$

$$\text{or, } (a-b-c)x^2 + \{a^2 - (a+b)c - b^2\}x - abc = 0,$$

which is arranged according to the dimensions of x.

81. If $\frac{a}{b}$ be any arithmetical fraction whatever, either proper or improper, then will $\frac{a}{b} + \frac{b}{a}$ be always greater than 2.

For, since by article (49), $a^2 + b^2$ is $> 2ab$, we have

$$\frac{a^2}{ab} + \frac{b^2}{ab} > \frac{2ab}{ab}, \text{ or } \frac{a}{b} + \frac{b}{a} > 2:$$

and consequently the least value that $\frac{a}{b} + \frac{b}{a}$ can admit of, will be 2, when a and b are equal.

82. *To prove the Rules for the Multiplication, Division, &c. of Fractions expressed decimally.*

Since *Decimal Fractions*, which are expressed in the same form as whole numbers, are equivalent to vulgar fractions having 10 and its powers for denominators, if P and Q comprise p and q decimal places respectively,

they will be expressed in the forms of vulgar fractions by

$$\frac{P}{10^p} \text{ and } \frac{Q}{10^q}.$$

Whence their product $= \frac{P}{10^p} \times \frac{Q}{10^q} = \frac{PQ}{10^{p+q}}$:

and from this it is concluded that the multiplication of Decimals is performed as in whole numbers, and that the product contains as many decimal places as are found in both the multiplicand and multiplier.

Also, their quotient $= \frac{P}{10^p} \div \frac{Q}{10^q} = \frac{P}{10^p} \times \frac{10^q}{Q} = \frac{P}{Q} \times \frac{10^q}{10^p}$, which will admit of three different forms according as p is greater than, equal to, or less than q.

(1) If p be greater than q, the quotient is $\frac{P}{Q} \times \frac{1}{10^{p-q}}$: from which we see that after the division is effected as in whole numbers, the quotient must comprise $p - q$ decimal places.

(2) If p be equal to q, the quotient will be $\frac{P}{Q}$, which is a whole number, if P be divisible by Q without a remainder.

(3) If p be less than q, the quotient is $\frac{P}{Q} \times 10^{q-p}$: and this shews that we must affix to the quotient obtained as in integers, a number of ciphers $= q - p$, and the result will be a whole number.

Again, the square of $\frac{P}{10^p} = \frac{P^2}{10^{2p}}$ has $2p$ decimals:

the cube of $\frac{P}{10^p} = \frac{P^3}{10^{3p}}$ has $3p$ decimals: &c.

and conversely: as in articles (40) and (42).

Upon this subject, see the Chapter on the *Theory of Decimals*, in the Author's *Arithmetic*.

83. Def. Two or more numbers are said to be *prime* to each other, when they have no integral common measure greater than the unit.

84. *If the product* ab *be divisible by* c, *and* b *and* c *be prime to each other: then will* c *be a divisor of* a.

For, since b and c are prime to each other, their common measure determined by the ordinary process must be 1: that is, we shall have an operation such as the following:

$$\begin{array}{lllll}
c\,)\,b\,(\,p & & & & \\
\quad\underline{pc} & & & & \\
 & d\,)\,c\,(\,q & & & \\
 & \quad\underline{qd} & & & \\
 & & e\,)\,d\,(\,r & & \\
 & & \quad\underline{re} & & \\
 & & & 1\,)\,e\,(\,e & \\
 & & & \quad\;\underline{e} &
\end{array}$$

from which, $b = pc + d$, $c = qd + e$, and $d = re + 1$:
whence, $ab = apc + ad$, $ac = aqd + ae$, and $ad = are + a$:
$\therefore\ ab - apc = ad$, $ac - aqd = ae$, and $ad - are = a$:
then, since ab is divisible by c, ad is divisible by c:
$\therefore\ ae$ is divisible by c, $\therefore\ a$ has c for a divisor:
and a similar proof will be applicable whatever be the requisite number of divisions, and also when b is less than c.

From this it is manifest that if c be prime to b and greater than a, ab is not divisible by c.

85. *If* a *be prime to* b, *there is no fraction equal to* $\frac{a}{b}$, *whose terms are not equimultiples of* a *and* b.

For, if possible, let $\frac{a}{b} = \frac{c}{d}$ where c and d are respectively less than a and b: then since $c = \frac{ad}{b}$, b must be a divisor of

ad: but a and b being prime to each other, b must be a divisor of d by the last article, which is absurd, since d is less than b by hypothesis: therefore it follows that $\frac{a}{b}$ is *irreducible*, and cannot be expressed in lower terms; and consequently, whenever $\frac{a}{b} = \frac{c}{d}$, c and d must be equimultiples of a and b.

86. *If two numbers be prime to each other, their sum or difference is prime to each of them.*

Let a and b denote the two numbers: and if possible, let a and $a \pm b$ have the common measure d, such that $a = pd$ and $a \pm b = qd$: then we shall have $b = \pm (q - p)\, d$, which proves that a and b have the common measure d, contrary to the hypothesis: similarly, of b and $a \pm b$.

A like process will shew that $a + b$ and $a - b$ are either prime to each other, or have the common measure 2.

87. *If one number be prime to each of two others, it is also prime to their product.*

Let a be prime to b and c, and if possible, let $a = pd$ and $bc = qd$: then, since b and c are prime to a or pd, they are each prime to d: also,

$$\text{since } bc = qd, \text{ we have } \frac{b}{d} = \frac{q}{c}:$$

and therefore, by article (85), c is a multiple of d: that is, a and c have a common measure d, which is absurd: therefore a is prime to bc.

88. *If the number* a *be prime to each of the numbers* b, c, d, &c. *it will also be prime to their product* bcd &c.

For, since a is prime to b and c, it is prime to their product bc: also, since a is prime to bc and d, it is prime to their product bcd: and so on.

Hence, also if a be prime to b and b, it is prime to b^2: similarly, a is prime to b^3, b^4, &c.: so that if $\frac{b}{a}$ be a fraction

in its lowest terms, $\frac{b^2}{a}$, $\frac{b^3}{a}$, $\frac{b^4}{a}$, &c. are also fractions in their lowest terms.

In the same manner, if a, b, c, &c. be each of them prime to each of the numbers A, B, C, &c., it may be shewn that abc &c. is prime to each of the quantities A, B, C, &c. and also to their product ABC &c.: and finally, that $\left(\frac{a}{A}\right)^2$, $\left(\frac{a}{A}\right)^3$, $\left(\frac{a}{A}\right)^4$, &c. are all irreducible fractions.

89. *If a number be divisible by two or more numbers which are prime to each other, it is also divisible by their product.*

For, let a be divisible by each of the numbers b, c, d, &c. which are prime to each other; then if $\frac{a}{b} = p$, or $a = pb$, we shall have $\frac{a}{c} = \frac{pb}{c} = \left(\frac{p}{c}\right) b$, where by article (84), $\frac{p}{c}$ must be a whole number: let this be q, so that $\frac{a}{c} = qb$, or $a = qbc$: therefore a is divisible by bc.

Again, $\frac{a}{d} = \frac{qbc}{d} = \left(\frac{q}{d}\right) bc$, where $\frac{q}{d}$ must be a whole number, because bc is prime to d by article (87): let this be r, so that $\frac{a}{d} = rbc$, or $a = rbcd$; and therefore a is divisible by bcd: and so on, whatever be the number of divisors prime to each other.

A similar method of reasoning would prove that the proposition is not true, when the divisors are composite numbers.

90. *To find under what circumstances, vulgar fractions are convertible into finite, or infinite decimals.*

Let $\frac{a}{b}$ be a proposed fraction in its lowest terms: then we have $\frac{a}{b} = \frac{1}{10^m}\left(\frac{a10^m}{b}\right)$, which will be a decimal consisting of m places when the division of $a10^m$ by b has been effected: but b being prime to a, it is evident that this division will terminate or not, according as b is a divisor of 10^m or not: and the only divisors of 10 being 2 and 5, it follows that when b is a divisor of 10^m, it must be of the form $2^p 5^q$: and consequently the fraction will be convertible into a terminating decimal when it is of the form $\frac{a}{2^p 5^q}$, but into a non-terminating decimal in all other cases.

91. COR. 1. Hence every fraction of the form $\frac{a}{2^p 5^q}$, will give a finite decimal consisting of a number of places equal to the greater of the numbers p and q.

For, if $p > q$, we have

$$\frac{a}{2^p 5^q} = \frac{a5^{-q}}{2^p} = \frac{a5^{p-q}}{2^p 5^p} = \frac{1}{10^p}(a5^{p-q}),$$

which therefore comprises p decimal places:

and, if $p < q$, we have

$$\frac{a}{2^p 5^q} = \frac{a2^{-p}}{5^q} = \frac{a2^{q-p}}{2^q 5^q} = \frac{1}{10^q}(a2^{q-p}),$$

which therefore consists of q places of decimals.

92. COR. 2. If a fraction in its lowest terms be convertible into a non-terminating decimal, the figures will *recur* in a certain order, and their number will always be less than the denominator.

For, if $\frac{a}{b}$ be converted into a decimal as indicated in article (90), it is evident that at every stage of the process, the remainder must be less than b, and consequently the number

of *different* remainders will be $b - 1$ at most: whence, in the progress of the division, we shall manifestly have to repeat the same operation upon the same symbols as before: and thus the figures in the quotient and remainders will continually recur in the same order.

Numbers whose digits thus recur in *periods*, are called *recurring* or *circulating* Decimals, for which see the *Arithmetic*.

93. We have seen that a fraction may sometimes be expressed by a mixed quantity; but it frequently happens that a fraction may be exhibited in a series which never terminates, as in Ex. (4) of article (34), by the following substitutions:

$$\frac{a}{1-b} = a + b\left(\frac{a}{1-b}\right) = a + b\left\{a + b\left(\frac{a}{1-b}\right)\right\}$$

$$= a + ab + b^2\left(\frac{a}{1-b}\right) = a + ab + b^2\left\{a + ab + b^2\left(\frac{a}{1-b}\right)\right\}$$

$$= a + ab + ab^2 + ab^3 + \&\text{c. } \textit{in infinitum}.$$

Here the symbolical result being true, whatever be the values of a and b, if we suppose $b = 1$, we shall have $\frac{a}{1-1}$, or $\frac{a}{0}$ for a symbolical representation of the series $a + a + a + \&\text{c.}$ *in infinitum*, whose value is indefinitely great, if a represent any finite numerical magnitude; that is, if infinite numerical magnitude be denoted by ∞, we shall have

$$\frac{a}{0} = \infty, \quad \therefore \frac{a}{\infty} = 0, \text{ and } 0 \times \infty = a;$$

which results may be enunciated generally in the following terms.

A finite quantity divided by zero, gives an infinite quotient: a finite quantity divided by an infinite quantity, gives zero for a quotient, and the product of zero and infinity may be finite.

CHAPTER V.

ALGEBRAICAL SURDS AND IMAGINARY QUANTITIES.

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94. Def. A Surd, or Irrational Quantity is here expressed in the same form as in Arithmetic, being merely the Indication of an Operation which cannot be effected in a finite number of symbols, and the notation adopted to characterize quantities of this description is explained in article (14). The Arithmetical processes, when applied to surds, depend materially upon the treatment of the fractional indices employed to express them, as will be seen in the following articles.

95. *To represent a rational quantity, in the form of a surd.*

Taking the symbol a to represent any rational quantity, we have $a = a^1 = a^{\frac{2}{2}} = \&c. = a^{\frac{m}{m}}$, the last form of which may be written $(a^{\frac{1}{m}})^m = (a^m)^{\frac{1}{m}} = \sqrt[m]{a^m}$, agreeably to the definitions of the operations implied.

Hence, also, a quantity in the form of a surd may in reality be equivalent to a rational quantity: thus,

$$\sqrt{\frac{a^3 - 3a^2 + 3a - 1}{a - 1}} = \sqrt{a^2 - 2a + 1} = a + 1\,.$$

96. *To represent the product of a rational and a surd factor, in the form of an entire surd.*

Let a represent the rational, and $\sqrt[m]{b}$ or $b^{\frac{1}{m}}$, the irrational factor: then, since

$$(ab^{\frac{1}{m}})^m = a^m \times (b^{\frac{1}{m}})^m = a^m b:$$

we have, by the indication of the reverse operation,

$$ab^{\frac{1}{m}} = (a^m b)^{\frac{1}{m}}, \text{ or } a\sqrt[m]{b} = \sqrt[m]{a^m b}.$$

Ex. Let the proposed quantity be $\frac{x+1}{x-1}\sqrt{\frac{x-1}{x+1}}$.

$$\text{This} = \sqrt{\left(\frac{x+1}{x-1}\sqrt{\frac{x-1}{x+1}}\right)^2} = \sqrt{\frac{(x+1)^2}{(x-1)^2}\cdot\frac{x-1}{x+1}}$$

$$= \sqrt{\frac{x+1}{x-1}}, \text{ or } = \left(\frac{x+1}{x-1}\right)^{\frac{1}{2}},$$

which is a much simpler form than the original.

97. *To reduce a surd to such a form, that its irrational factor may be the simplest possible.*

This is merely the converse of the last article, for

$$\sqrt[m]{a^m b} = \sqrt[m]{a^m}\sqrt[m]{b} = a\sqrt[m]{b}:$$

$$\text{similarly, } (a^{m+n}x)^{\frac{1}{m}} = a^{\frac{m+n}{m}}x^{\frac{1}{m}} = a^{1+\frac{n}{m}}x^{\frac{1}{m}}$$

$$= a \times a^{\frac{n}{m}}x^{\frac{1}{m}} = a \times (a^n x)^{\frac{1}{m}} = a\sqrt[m]{a^n x}.$$

Ex. Reduce $\sqrt{27a^3x^5}$ to its simplest form.

$$\text{Here, } \sqrt{27a^3x^5} = \sqrt{9a^2x^4 \times 3ax} = \sqrt{9a^2x^4} \times \sqrt{3ax}$$

$$= 3ax^2\sqrt{3ax}.$$

This article will be found useful in combining surds together, by the operations of Addition and Subtraction.

98. *To reduce two or more surds to others, having a common index.*

This will of course be effected by the ordinary treatment of their indices: thus, if the quantities be $a^{\frac{m}{n}}$ and $b^{\frac{p}{q}}$, we have

$$a^{\frac{m}{n}} = a^{\frac{m}{n}\times\frac{q}{q}} = a^{\frac{mq}{nq}}, \text{ and } b^{\frac{p}{q}} = b^{\frac{p}{q}\times\frac{n}{n}} = b^{\frac{np}{nq}}:$$

and these results are respectively equivalent to $(a^{mq})^{\frac{1}{nq}}$ and $(b^{np})^{\frac{1}{nq}}$, which have the common index $\dfrac{1}{nq}$.

Ex. Reduce $(ax)^{\frac{1}{2}}$ and $(bx^2)^{\frac{1}{3}}$, so as to have a common index.

$$\text{Here,}\quad (ax)^{\frac{1}{2}} = a^{\frac{1}{2}}x^{\frac{1}{2}} = a^{\frac{3}{6}}x^{\frac{3}{6}} = (a^3x^3)^{\frac{1}{6}}:$$

$$\text{and}\quad (bx^2)^{\frac{1}{3}} = b^{\frac{1}{3}}x^{\frac{2}{3}} = b^{\frac{2}{6}}x^{\frac{4}{6}} = (b^2x^4)^{\frac{1}{6}}:$$

the indices being here transformed so as to have the least common denominator 6.

The chief use of this article will be found in the reduction of the operations of Multiplication and Division in surds.

99. *To find the sum and difference of two surds.*

The sum and difference being indicated by the proper algebraical signs, the incorporation may then be effected as far as it is possible, by means of article (97).

Ex. 1. Find the sum and difference of $\sqrt{4ax^2}$ and $3x\sqrt{9a}$.

$$\begin{aligned}\text{Here, the sum } &= \sqrt{4ax^2} + 3x\sqrt{9a}\\ &= \sqrt{4x^2 \times a} + 3x\sqrt{9 \times a}\\ &= 2x\sqrt{a} + 9x\sqrt{a}\\ &= (2x + 9x)\sqrt{a} = 11x\sqrt{a}:\end{aligned}$$

$$\text{the difference} = (2x - 9x)\sqrt{a} = -7x\sqrt{a}.$$

Ex. 2. Simplify as much as possible, the expression

$$9x\sqrt[3]{2a^5x^2} - 8a\sqrt[3]{2a^2x^5} + 2ax\sqrt[3]{2a^2x^2}.$$

$$\text{Here,}\quad 9x\sqrt[3]{2a^5x^2} = 9x\sqrt[3]{a^3 \times 2a^2x^2} = 9ax\sqrt[3]{2a^2x^2}:$$

$$8a\sqrt[3]{2a^2x^5} = 8a\sqrt[3]{x^3 \times 2a^2x^2} = 8ax\sqrt[3]{2a^2x^2}:$$

therefore the proposed expression is equivalent to

$$9ax\sqrt[3]{2a^2x^2} - 8ax\sqrt[3]{2a^2x^2} + 2ax\sqrt[3]{2a^2x^2}$$

$$= (9ax - 8ax + 2ax)\sqrt[3]{2a^2x^2} = 3ax\sqrt[3]{2a^2x^2}.$$

100. *To find the product and quotient of two surds.*

The product and quotient being expressed by means of the requisite algebraical signs, it remains only to reduce the results as much as possible, according to the method of article (98).

101. Before we proceed to exemplify the last article, we will establish from first principles, the two equalities adopted in article (15): namely,

$$a^{\frac{p}{q}} \times a^{\frac{r}{s}} = a^{\frac{p}{q}+\frac{r}{s}},$$

$$a^{\frac{p}{q}} \div a^{\frac{r}{s}} = a^{\frac{p}{q}-\frac{r}{s}},$$

where p, q, r and s are positive whole numbers.

Assume $a^{\frac{p}{q}} = x$, and $a^{\frac{r}{s}} = y$: then, from the nature of the operations indicated, we have

$$(a^{\frac{p}{q}})^q = x^q, \text{ or } a^p = x^q: \text{ and } (a^{\frac{r}{s}})^s = y^s, \text{ or } a^r = y^s:$$

whence, $a^{ps} = x^{qs}$, and $a^{qr} = y^{qs}$, by (5) of article (13):

$\therefore$ $a^{ps} \times a^{qr} = x^{qs} \times y^{qs}$, or $a^{ps+qr} = (xy)^{qs}$, by (1) of article (13):

consequently, $xy = a^{\frac{ps+qr}{qs}}$, by article (14):

that is, $a^{\frac{p}{q}} \times a^{\frac{r}{s}} = a^{\frac{ps}{qs}+\frac{qr}{qs}} = a^{\frac{p}{q}+\frac{r}{s}}$.

Similarly, $a^{ps} \div a^{qr} = x^{qs} \div y^{qs}$, or $a^{ps-qr} = \left(\frac{x}{y}\right)^{qs}$, by (2) of article (13):

whence, $\frac{x}{y} = a^{\frac{ps-qr}{qs}}$, by article (14):

that is, $a^{\frac{p}{q}} \div a^{\frac{r}{s}} = a^{\frac{ps}{qs}-\frac{qr}{qs}} = a^{\frac{p}{q}-\frac{r}{s}}$.

Hence, the results of the Multiplication and Division of algebraical symbols are always expressed in the same forms, whether the indices be positive or negative, integral or fractional.

Ex. 1. The product of $ax^{\frac{1}{2}}$ and $bx^{\frac{1}{3}}$ is

$$abx^{\frac{1}{2}+\frac{1}{3}} = abx^{\frac{3}{6}+\frac{2}{6}} = abx^{\frac{5}{6}}.$$

Ex. 2. The quotient of $\sqrt{ax^3}$ by $\sqrt{bx}$ is expressed by

$$\frac{a^{\frac{1}{2}}x^{\frac{3}{2}}}{b^{\frac{1}{2}}x^{\frac{1}{2}}} = \left(\frac{a}{b}\right)^{\frac{1}{2}} x^{\frac{3}{2}-\frac{1}{2}} = \left(\frac{a}{b}\right)^{\frac{1}{2}} x = \frac{x}{b}\sqrt{ab}.$$

Ex. 3. Multiply $x - \sqrt{xy} + y$ by $\sqrt{x} + \sqrt{y}$.

Here, expressing the surds by means of fractional indices, we have

$$\begin{array}{l}
x - x^{\frac{1}{2}}y^{\frac{1}{2}} + y \\
x^{\frac{1}{2}} + y^{\frac{1}{2}} \\
\hline
x^{\frac{3}{2}} - x y^{\frac{1}{2}} + x^{\frac{1}{2}}y \\
\qquad\quad x y^{\frac{1}{2}} - x^{\frac{1}{2}}y + y^{\frac{3}{2}} \\
\hline
x^{\frac{3}{2}} + y^{\frac{3}{2}}
\end{array}$$

the different factors involving x and y being combined by the addition of their indices respectively.

Ex. 4. By a similar process, we shall have

$$\begin{array}{l}
a^{\frac{5}{2}} + a^2 b^{\frac{1}{3}} + a^{\frac{3}{2}} b^{\frac{2}{3}} + a\, b + a^{\frac{1}{2}} b^{\frac{4}{3}} + b^{\frac{5}{3}} \\
a^{\frac{1}{2}} - b^{\frac{1}{3}} \\
\hline
a^3 + a^{\frac{5}{2}} b^{\frac{1}{3}} + a^2 b^{\frac{2}{3}} + a^{\frac{3}{2}} b + a\, b^{\frac{4}{3}} + a^{\frac{1}{2}} b^{\frac{5}{3}} \\
\qquad - a^{\frac{5}{2}} b^{\frac{1}{3}} - a^2 b^{\frac{2}{3}} - a^{\frac{3}{2}} b - a\, b^{\frac{4}{3}} - a^{\frac{1}{2}} b^{\frac{5}{3}} - b^2 \\
\hline
\end{array}$$

Ex. 5. Divide $a+b-c+2\sqrt{ab}$ by $\sqrt{a}+\sqrt{b}-\sqrt{c}$.

Here, using fractional indices, we shall have

$$a^{\frac{1}{2}}+b^{\frac{1}{2}}-c^{\frac{1}{2}}\,)\,a+b-c+2a^{\frac{1}{2}}b^{\frac{1}{2}}\,(\,a^{\frac{1}{2}}+b^{\frac{1}{2}}+c^{\frac{1}{2}}$$

$$a+a^{\frac{1}{2}}b^{\frac{1}{2}}-a^{\frac{1}{2}}c^{\frac{1}{2}}$$

$$a^{\frac{1}{2}}b^{\frac{1}{2}}+a^{\frac{1}{2}}c^{\frac{1}{2}}+b$$

$$a^{\frac{1}{2}}b^{\frac{1}{2}}+b-b^{\frac{1}{2}}c^{\frac{1}{2}}$$

$$a^{\frac{1}{2}}c^{\frac{1}{2}}+b^{\frac{1}{2}}c^{\frac{1}{2}}-c$$

$$a^{\frac{1}{2}}c^{\frac{1}{2}}+b^{\frac{1}{2}}c^{\frac{1}{2}}-c$$

and in this, the index of the first letter in the divisor is subtracted from that of the first letter in the dividend, just as if the indices were whole numbers.

Ex. 6. To divide a^3-b^3 by $a^{\frac{3}{4}}+b^{\frac{3}{4}}$, we have

$$a^{\frac{3}{4}}+b^{\frac{3}{4}}\,)\,a^{\frac{12}{4}}-b^{\frac{12}{4}}\,(\,a^{\frac{9}{4}}-a^{\frac{6}{4}}b^{\frac{3}{4}}+a^{\frac{3}{4}}b^{\frac{6}{4}}-b^{\frac{9}{4}}$$

$$a^{\frac{12}{4}}+a^{\frac{9}{4}}b^{\frac{3}{4}}$$

$$-a^{\frac{9}{4}}b^{\frac{3}{4}}-b^{\frac{12}{4}}$$

$$-a^{\frac{9}{4}}b^{\frac{3}{4}}-a^{\frac{6}{4}}b^{\frac{6}{4}}$$

$$a^{\frac{6}{4}}b^{\frac{6}{4}}-b^{\frac{12}{4}}$$

$$a^{\frac{6}{4}}b^{\frac{6}{4}}+a^{\frac{3}{4}}b^{\frac{9}{4}}$$

$$-a^{\frac{3}{4}}b^{\frac{9}{4}}-b^{\frac{12}{4}}$$

$$-a^{\frac{3}{4}}b^{\frac{9}{4}}-b^{\frac{12}{4}}$$

wherein we have used all the indices as fractions having the common denominator 4; and by simplifying the indices, the quotient becomes $a^{\frac{9}{4}}-a^{\frac{3}{2}}b^{\frac{3}{4}}+a^{\frac{3}{4}}b^{\frac{3}{2}}-b^{\frac{9}{4}}$.

102. *To express the powers and roots of a surd.*

Let $a^{\frac{p}{q}}$ be a proposed quantity in the form of a surd: then

$$\text{its square} = a^{\frac{p}{q}} \times a^{\frac{p}{q}} = a^{\frac{p}{q}+\frac{p}{q}} = a^{\frac{2p}{q}}:$$

$$\text{its cube} = a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times a^{\frac{p}{q}} = a^{\frac{2p}{q}} \times a^{\frac{p}{q}} = a^{\frac{2p}{q}+\frac{p}{q}} = a^{\frac{3p}{q}}: \text{ \&c.}$$

and continuing the process by the last article, we have the r^{th} power of $a^{\frac{p}{q}}$, or $(a^{\frac{p}{q}})^r = a^{\frac{pr}{q}}$.

Again, the s^{th} root of $a^{\frac{p}{q}}$ will be expressed by $a^{\frac{p}{qs}}$, because the s^{th} power of $a^{\frac{p}{qs}} = a^{\frac{ps}{qs}} = a^{\frac{p}{q}}$, by the preceding.

Also, the $\dfrac{r}{s}$th power of $a^{\frac{p}{q}}$ will be expressed by $a^{\frac{pr}{qs}}$:

$$\text{for, if } a^{\frac{p}{q}} = x, \text{ then } a^p = x^q, \text{ and } a^{pr} = x^{qr}:$$

$$\therefore\ a^{\frac{pr}{qs}} = x^{\frac{qr}{qs}} = x^{\frac{r}{s}} = (a^{\frac{p}{q}})^{\frac{r}{s}}, \text{ or } (a^{\frac{p}{q}})^{\frac{r}{s}} = a^{\frac{pr}{qs}}.$$

Hence, the results of involution and evolution are expressed in the same forms, whether the indices be positive or negative, integral or fractional.

Ex. 1. The square of $(ax^3)^{\frac{1}{4}}$ or of $a^{\frac{1}{4}}x^{\frac{3}{4}}$, is $a^{\frac{1}{2}}x^{\frac{3}{2}}$, obtained by doubling the index of each factor.

Ex. 2. The cube root of $b^3x^{\frac{3}{4}}$ will be $bx^{\frac{1}{4}}$, which is had by dividing each index by 3.

Ex. 3. The square and cube of $a - bx^{\frac{1}{2}}$ will be found to be $a^2 - 2abx^{\frac{1}{2}} + b^2x$, and $a^3 - 3a^2bx^{\frac{1}{2}} + 3ab^2x - b^3x^{\frac{3}{2}}$ by actual multiplication; and these operations reversed give $a - bx^{\frac{1}{2}}$ for the square and cube root of the latter quantities.

BINOMIAL SURDS.

103. Def. The irrational quantities most commonly met with in the Applications of Arithmetic and Algebra to *Philosophical Enquiries*, present themselves under the forms $a \pm \sqrt{b}$, and $\sqrt{a} \pm \sqrt{b}$, and are generally termed ***Binomial Surds.***

We will here exemplify their treatment in the operations of Division, and the Extraction of the Square Root only, as the rest possess little or nothing peculiar.

(1) If we wish to divide $15 + 7\sqrt{3}$ by $9 + 5\sqrt{3}$, and proceed by the rule for ordinary division, we shall soon see that the quotient thus found, will be an interminable quantity: but by multiplying the numerator and denominator of the fraction by $9 - 5\sqrt{3}$, we have the quotient $= \dfrac{15 + 7\sqrt{3}}{9 + 5\sqrt{3}}$

$$= \frac{(15 + 7\sqrt{3})(9 - 5\sqrt{3})}{(9 + 5\sqrt{3})(9 - 5\sqrt{3})} = \frac{30 - 12\sqrt{3}}{81 - 75} = \frac{30 - 12\sqrt{3}}{6}$$

$$= 5 - 2\sqrt{3}.$$

A similar process may be adopted in every other case of this kind, and also when each of the terms of the dividend and divisor is a surd: thus,

$$\frac{3\sqrt{5} + \sqrt{3}}{\sqrt{5} - \sqrt{3}} = \frac{(3\sqrt{5} + \sqrt{3})(\sqrt{5} + \sqrt{3})}{(\sqrt{5} - \sqrt{3})(\sqrt{5} + \sqrt{3})} = \frac{18 + 4\sqrt{15}}{5 - 3}$$

$$= 9 + 2\sqrt{15}.$$

(2) It is evident that when these quotients are reduced to their ultimate forms, the approximate arithmetical computation of their values will be greatly facilitated: thus, by finding only the square root of 15 to four places of decimals, we have the value of $9 + 2\sqrt{15} = 16.7458$ &c.; whereas, the same result could not have been obtained from $\dfrac{3\sqrt{5} + \sqrt{3}}{\sqrt{5} - \sqrt{3}}$ without the extraction of the square roots of 3 and 5 to several places of decimals, and the subsequent operation of long division. This reduction is effected by the multiplication of the numerator and denominator by a factor formed by changing the sign of either of the terms of the denominator.

(3) In the same way, $\dfrac{a+\sqrt{b}}{c+\sqrt{d}} = \dfrac{(a+\sqrt{b})(c-\sqrt{d})}{(c+\sqrt{d})(c-\sqrt{d})}$
$= \dfrac{ac - a\sqrt{d} + c\sqrt{b} - \sqrt{bd}}{c^2 - d}$, which is now expressed in the form of a fraction with a *rational* denominator, and is equivalent to

$$\frac{ac}{c^2-d} - \frac{a}{c^2-d}\sqrt{d} + \frac{c}{c^2-d}\sqrt{b} - \frac{1}{c^2-d}\sqrt{bd}:$$

but the labour of arithmetical computation is scarcely diminished by this transformation.

(4) The square of an algebraical *binomial* is generally expressed by a *trinomial*, whose terms are all distinct from each other: and there is consequently not much difficulty in ascending from the square, to the root from which it was derived, as appears from article (38).

If however the root be a binomial surd, as $\sqrt{2}+1$, its square will be $2+2\sqrt{2}+1$, or $3+2\sqrt{2}$, which is also a binomial surd: and consequently the ordinary rule for reversing the operation, or for extracting the square root, will not be available unless the binomial surd be replaced by its proper corresponding trinomial quantity, which in many cases it may be no easy matter to do.

Thus, the square of $\sqrt{5}-\sqrt{2} = 5 - 2\sqrt{10} + 2$, and the operation might easily be reversed by means of the result in its present form: but by incorporating the terms as much as possible, we obtain $7 - 2\sqrt{10}$, which affords but little trace of the root.

On these grounds certain steps are usually pursued, which it shall be our object to detail in the following articles.

104. *The product of two quadratic surds will be a surd, unless one of them is some rational multiple, part or parts of the other.*

For, if possible, let $\sqrt{x}\times\sqrt{y} = m$: then squaring both sides, we have $xy = m^2$, and $\therefore\ y = \dfrac{m^2}{x} = \dfrac{m^2}{x^2}x$: whence it

follows that $\sqrt{y} = \frac{m}{x}\sqrt{x}$; or, $\sqrt{y}$ is some rational multiple, part or parts of $\sqrt{x}$, which is contrary to the hypothesis: and consequently the product of two quadratic surds is a surd, with these restrictions.

105. *One quadratic surd cannot be made up of a rational quantity and another quadratic surd, nor of two other quadratic surds, connected by the operation of addition or subtraction.*

For, if possible, let $\sqrt{a} = x \pm \sqrt{y}$: then by squaring both members of the equality, we have

$$a = x^2 \pm 2x\sqrt{y} + y,$$

which gives $\sqrt{y} = \pm\frac{a - x^2 - y}{2x}$; or a surd equivalent to a rational quantity, which is impossible.

Again, if $\sqrt{a} = \sqrt{x} \pm \sqrt{y}$, we shall have

$$a = x \pm 2\sqrt{xy} + y,$$

from which we find $\sqrt{xy} = \pm\frac{1}{2}(a - x - y)$, or a surd equivalent to a rational quantity, as appears from article (104), which is impossible.

From this it follows that the equalities,

$$\sqrt{a} = x \pm \sqrt{y}, \text{ and } \sqrt{a} = \sqrt{x} \pm \sqrt{y},$$

can have no existence, provided the values of a, x, and y be any numbers whatever.

106. Cor. Hence, whenever an equality appears expressed in the form $a + \sqrt{b} = x + \sqrt{y}$, we must have

$$a = x, \text{ and } \sqrt{b} = \sqrt{y}.$$

For, if not, let $a = x + \alpha$: then will

$$x + \alpha + \sqrt{b} = x + \sqrt{y}, \text{ and } \therefore\ \alpha + \sqrt{b} = \sqrt{y},$$

which has just been said to have no existence, unless $\alpha = 0$; therefore $a = x$, and $\sqrt{b} = \sqrt{y}$.

From this, the practical conclusion is, that if

$$a + \sqrt{b} = x + \sqrt{y}, \text{ then will } a - \sqrt{b} = x - \sqrt{y}.$$

107. *To extract the square root of a binomial surd, one of whose terms is a rational quantity, and the other a quadratic surd.*

Let $a + \sqrt{b}$ represent the proposed surd, and assume $\sqrt{a+\sqrt{b}} = \sqrt{u} + \sqrt{v}$, which gives $a + \sqrt{b} = u + v + 2\sqrt{uv}$;

$$\therefore \text{ by article (106), } u + v = a, \text{ and } 2\sqrt{uv} = \sqrt{b};$$

from which we have to find u and v in terms of a and b.

Now, squaring both members of these equalities, and subtracting, we have

$$\begin{aligned} u^2 + 2uv + v^2 &= a^2 \\ 4uv \qquad &= b \\ \hline u^2 - 2uv + v^2 &= a^2 - b: \end{aligned}$$

which, by the extraction of the square root, gives

$$u - v = \pm\sqrt{a^2 - b}.$$

Thus, we have $u + v = a$:

$$\text{and } u - v = \sqrt{a^2 - b}:$$

from which, in accordance with the purport of article (44), we find

$$2u = a + \sqrt{a^2 - b}, \text{ and } u = \frac{a + \sqrt{a^2 - b}}{2}:$$

$$2v = a - \sqrt{a^2 - b}, \text{ and } v = \frac{a - \sqrt{a^2 - b}}{2}:$$

$\therefore$ the square root of $a + \sqrt{b}$ will be expressed by

$$\sqrt{\frac{a + \sqrt{a^2 - b}}{2}} + \sqrt{\frac{a - \sqrt{a^2 - b}}{2}}.$$

In the same manner, we should assume the square root of $a - \sqrt{b}$ to be represented by $\sqrt{u} - \sqrt{v}$: and a similar result would be obtained, differing from the former only in the sign of its second term.

The results of this article being generally expressed by

$$\sqrt{a \pm \sqrt{b}} = \sqrt{\frac{a + \sqrt{a^2 - b}}{2}} \pm \sqrt{\frac{a - \sqrt{a^2 - b}}{2}},$$

will always be symbolically correct, and are capable of immediate verification. It will be observed, however, that when the symbols are general, one complex quadratic surd has been resolved into two others still more complicated in their forms: and such a result can never be of any practical use in computation.

Whenever the proposed surds are numerical magnitudes, and such a relation subsists among their parts, that $a^2 - b$ is equal to some complete square c^2, we have the much simplified form

$$\sqrt{a \pm \sqrt{b}} = \sqrt{\frac{a + c}{2}} \pm \sqrt{\frac{a - c}{2}},$$

where a complex surd has been resolved into two simple surds, one of which may be equivalent to an integer, provided either $\frac{1}{2}(a + c)$ or $\frac{1}{2}(a - c)$ be a complete square.

This article is of no practical use whatever in Arithmetical Computation, unless the *Criterion* $a^2 - b = c^2$, be satisfied: but whenever it is applicable, the diminution of the number of figures employed will be considerable.

Ex. 1. Extract the square root of $4+2\sqrt{3}$.

Here, $a^2-b=16-12=4=2^2=c^2$, and the criterion is satisfied:

let $\therefore\ \sqrt{4+2\sqrt{3}}=\sqrt{u}+\sqrt{v}$; $\therefore\ 4+2\sqrt{3}=u+v+2\sqrt{uv}$:

$$\text{whence, } u+v=4, \text{ and } 2\sqrt{uv}=2\sqrt{3}:$$

$$\begin{aligned} \text{also, } u^2+2uv+v^2 &= 16 \\ 4uv &= 12 \\ \hline \therefore\ u^2-2uv+v^2 &= 4 \end{aligned}$$

and the extraction of the square root gives $u-v=2$:

thus, we have $\left.\begin{aligned} u+v&=4 \\ \text{and } u-v&=2\end{aligned}\right\}$, from which we obtain immediately

$$2u=6,\ u=3,\ \sqrt{u}=\sqrt{3}:\ 2v=2,\ v=1,\ \sqrt{v}=1:$$

and the required square root is $\sqrt{3}+1$.

In the equality, $\sqrt{4+2\sqrt{3}}=\sqrt{3}+1$, we observe that the complex surd of the former member is equivalent to the simple surd and integer of the latter: and consequently in the computation of the approximate value of the quantity proposed, it is clear that a much less number of figures will be required for the latter form than for the former.

Exactly in the same manner, $\sqrt{4-2\sqrt{3}}=\sqrt{3}-1$.

Ex. 2. Find the square root of the algebraical binomial surd $x-2\sqrt{x-1}$.

Assume $\sqrt{x-2\sqrt{x-1}}=\sqrt{u}-\sqrt{v}$, which leads to

$$x-2\sqrt{x-1}=u+v-2\sqrt{uv},\ u+v=x,$$

$$\text{and } 2\sqrt{uv}=2\sqrt{x-1};$$

$$\begin{aligned} \text{whence, we have, } u^2+2uv+v^2 &= x^2 \\ 4uv &= 4x-4 \\ \hline \therefore\ u^2-2uv+v^2 &= x^2-4x+4 \end{aligned}$$

and by the extraction of the square root, we find $u-v=x-2$: so that the equalities

$$u+v=x, \text{ and } u-v=x-2, \text{ enable us to obtain}$$

$$\sqrt{u}=\sqrt{x-1}, \text{ and } \sqrt{v}=1:$$

and therefore the required square root is $\sqrt{x-1}-1$.

In both these examples, the requisite substitutions in the general formula would have produced the same results: but in practice it will generally be found more convenient to go through the operations at length.

108. We will further illustrate the importance of the last article, by the following instance of the reduction of a very complicated expression to a very simple form:

$$\frac{2+\sqrt{3}}{\sqrt{2}+\sqrt{2+\sqrt{3}}}+\frac{2-\sqrt{3}}{\sqrt{2}-\sqrt{2-\sqrt{3}}}$$

$$=\frac{2\sqrt{2}+\sqrt{6}}{2+\sqrt{4+2\sqrt{3}}}+\frac{2\sqrt{2}-\sqrt{6}}{2-\sqrt{4-2\sqrt{3}}}$$

$$=\frac{2\sqrt{2}+\sqrt{6}}{2+(\sqrt{3}+1)}+\frac{2\sqrt{2}-\sqrt{6}}{2-(\sqrt{3}-1)}=\frac{2\sqrt{2}+\sqrt{6}}{3+\sqrt{3}}+\frac{2\sqrt{2}-\sqrt{6}}{3-\sqrt{3}}$$

$$=\sqrt{\frac{2}{3}}\left\{\frac{2+\sqrt{3}}{\sqrt{3}+1}+\frac{2-\sqrt{3}}{\sqrt{3}-1}\right\}=\sqrt{\frac{2}{3}}\left(\frac{2\sqrt{3}}{2}\right)=\sqrt{2}.$$

Similarly, $$\frac{2+\sqrt{3}}{\sqrt{2}+\sqrt{2+\sqrt{3}}}-\frac{2-\sqrt{3}}{\sqrt{2}-\sqrt{2-\sqrt{3}}}=\sqrt{\frac{2}{3}}.$$

109. When the criterion is not satisfied, it sometimes happens that the square root may be extracted, after divesting the terms of a common factor.

Ex. If the proposed surd be $4+\sqrt{18}$, wherein $a^2-b=16-18=-2$, we observe that

$$4+\sqrt{18}=4+\sqrt{9\times 2}=4+3\sqrt{2}=\sqrt{2}\,(3+2\sqrt{2}),$$

the latter factor of which evidently satisfies the criterion:

$$\therefore \text{ the square root of } 4+\sqrt{18}=\sqrt[4]{2}\sqrt{3+2\sqrt{2}}$$
$$=\sqrt[4]{2}(\sqrt{2}+1), \text{ by the ordinary process,}$$
$$=\sqrt[4]{2}(\sqrt[4]{4}+1)=\sqrt[4]{8}+\sqrt[4]{2}.$$

The square root of $4+\sqrt{18}$ is here expressed in the form of a binomial surd, by $\sqrt[4]{8}+\sqrt[4]{2}$: but in this case, no advantage in the computation of its arithmetical value is obtained by the operation.

In the same manner, the square root of a binomial surd of the form $\sqrt{a}\pm\sqrt{b}$, may sometimes be exhibited in the form of a binomial surd: thus, the square root of

$$\sqrt{32}-\sqrt{24} = \text{the square root of}$$
$$\sqrt{2}(4-2\sqrt{3})=\sqrt[4]{2}\sqrt{4-2\sqrt{3}}$$
$$=\sqrt[4]{2}(\sqrt{3}-1)=\sqrt[4]{2}(\sqrt[4]{9}-1)=\sqrt[4]{18}-\sqrt[4]{2}.$$

110. An assumption similar to that made in article (107), will frequently enable us to extract the square root of a surd quantity comprising more terms than two: thus,

let $\sqrt{6+2\sqrt{2}+2\sqrt{3}+2\sqrt{6}}=\sqrt{u}+\sqrt{v}+\sqrt{w}$:

then, $6+2\sqrt{2}+2\sqrt{3}+2\sqrt{6}$

$$=u+v+w+2\sqrt{uv}+2\sqrt{uw}+2\sqrt{vw}:$$

whence, we have $u+v+w=6$,

$$2\sqrt{uv}=2\sqrt{2},\ 2\sqrt{uw}=2\sqrt{3}, \text{ and } 2\sqrt{vw}=2\sqrt{6}:$$

$\therefore\ uv=2,\ uw=3,\ vw=6$, and $\therefore\ u^2v^2w^2=36$, or $uvw=6$:

$$\therefore\ u=\frac{uvw}{vw}=\frac{6}{6}=1,\ v=\frac{uvw}{uw}=\frac{6}{3}=2, \text{ and } w=\frac{uvw}{uv}=\frac{6}{2}=3:$$

and the required square root is $1+\sqrt{2}+\sqrt{3}$, since these values of u, v and w satisfy also the condition $u+v+w=6$.

111. *To extract, when possible, the cube root of a binomial surd, one of whose terms is a rational quantity, and the other a quadratic surd.*

$$\text{Let } \sqrt[3]{a+\sqrt{b}} = (x+\sqrt{y})\sqrt[3]{z}:$$

then, according to the principle of article (106), we shall have

$$\sqrt[3]{a-\sqrt{b}} = (x-\sqrt{y})\sqrt[3]{z}:$$

$\therefore$ multiplying together the corresponding members, we obtain

$$\sqrt[3]{a^2-b} = (x^2-y)\,z^{\frac{2}{3}}, \text{ or } \sqrt[3]{(a^2-b)\,z} = (x^2-y)\,z:$$

whence, if z be so assumed that $(a^2-b)\,z$ is a perfect cube c^3, we have $c = (x^2-y)\,z$, from which $x^2 - y = \frac{c}{z}$:

$$\text{also, } a+\sqrt{b} = (x^3 + 3x^2\sqrt{y} + 3xy + y\sqrt{y})\,z,$$

furnishes the equalities

$$a = (x^3+3xy)\,z, \text{ and } \sqrt{b} = (3x^2+y)\,z\sqrt{y};$$

for determining the value of x or y.

The use of this article will be made clear by means of the following examples.

Ex. 1. Extract the cube root of $20 + 14\sqrt{2}$.

$$\text{Here, } \sqrt[3]{20+14\sqrt{2}} = (x+\sqrt{y})\sqrt[3]{z},$$

$$\text{and } \sqrt[3]{20-14\sqrt{2}} = (x-\sqrt{y})\sqrt[3]{z}:$$

$$\text{whence, } \sqrt[3]{8z} = (x^2-y)\,z, \text{ and } \therefore\ z = 1, \text{ and } y = x^2 - 2:$$

$$\text{also, } 20 + 14\sqrt{2} = x^3 + 3x^2\sqrt{y} + 3xy + y\sqrt{y}, \text{ gives}$$

$$20 = x^3 + 3xy = x^3 + 3x\,(x^2-2) = 4x^3 - 6x,$$

from which, x is immediately found by trial to be 2:

$$\text{that is, } x = 2, \text{ and } \therefore\ y = x^2 - 2 = 4 - 2 = 2:$$

and therefore the cube root of $20 + 14\sqrt{2}$ is $2 + \sqrt{2}$.

Ex. 2. Required the cube root of $11 - 5\sqrt{7}$.

$$\text{Here, } \sqrt[3]{11-5\sqrt{7}} = (x-\sqrt{y})z^{\frac{1}{3}},$$

$$\text{and } \sqrt[3]{11+5\sqrt{7}} = (x+\sqrt{y})z^{\frac{1}{3}}:$$

$$\therefore \sqrt[3]{-54z} = (x^2 - y)z, \quad \therefore z = \tfrac{1}{2}, \text{ and } y = x^2 + 6:$$

also, $11 - 5\sqrt{7} = (x^3 - 3x^2\sqrt{y} + 3xy - y\sqrt{y})\frac{1}{2}$, gives

$$-5\sqrt{7} = -\tfrac{1}{2}(3x^2 + y)\sqrt{y}, \text{ or } 10\sqrt{7} = (3x^2 + y)\sqrt{y}:$$

from which we find $y = 7$, and therefore $x = 1$: whence, the cube root required is $\dfrac{1 - \sqrt{7}}{\sqrt[3]{2}}$.

Whenever x is a whole number, a few trials will always give its value, but in other cases the process would be tedious, and it is better to find y as in the last example: or recourse must be had, as for the higher roots, to the principles explained in the chapter on the Binomial Theorem.

TRANSFORMATION OF SURDS.

112. A Surd may frequently be converted into a series, by a continuation of the operation indicated to any extent we please: but it is seldom possible to discover the law according to which the successive terms are formed.

Thus, by the ordinary rule for the extraction of the square root, we shall find

$$\sqrt{a^2 + x^2} = a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \text{\&c. } \textit{in infinitum}:$$

but this result will be of little use in arithmetical computation, unless x be very small compared with a.

113. In extracting the square root of a number, which according to article (40), has $2n + 1$ figures in its root, when $n + 1$ figures have been found, the remaining n figures may be obtained by dividing by the corresponding *trial* divisor.

For, if $a + b$ be the root consisting of $2n + 1$ figures, and a consisting of $n + 1$ figures followed by n ciphers has been found, then the remainder is $2ab + b^2$, which divided by the trial divisor $2a$ gives the quotient $b + \dfrac{b^2}{2a}$, differing

from b by the quantity $\frac{b^2}{2a}$: but, since b consists of n figures, b^2 cannot comprise more than $2n$ figures by article (40), whilst a comprises $2n + 1$ figures, and therefore $\frac{b^2}{2a}$ will be a proper fraction, which being neglected, will manifestly occasion no alteration in the root: and thus, the remaining n figures of the root will be obtained by division only.

Ex. To find the square root of 2, to seven places of figures.

$$
\begin{array}{rl}
 & \dot{2}.\dot{0}0\dot{0}0\dot{0}0\,(1.414213\ \&c. \\
 & 1 \\
\hline
24) & 100 \\
 & 96 \\
\hline
281) & 400 \\
 & 281 \\
\hline
2824) & 11900 \\
 & 11296 \\
\hline
2828) & 6040 \\
 & 5656 \\
\hline
 & 3840 \\
 & 2828 \\
\hline
 & 10120 \\
 & 8484 \\
\hline
 & 1636 \\
\hline
\end{array}
$$

that is, $\sqrt{2} = 1.414213$ &c., which is correct to the last place of decimals, and the three figures on the right hand have been obtained by division, from using the trial divisor 2828.

This article is the foundation of the observation at the end of article (159), of the Author's *Arithmetic*.

114. To approximate to the square root of a number, by means of fractions which are more and more nearly equal to its true value.

Let n represent the proposed number, and suppose a^2 to be the greatest square number contained in it, and b a quantity such that

$$n = (a + b)^2 = a^2 + 2ab + b^2:$$

then, since b is necessarily a proper fraction, we shall have

$$n = a^2 + 2ab \text{ nearly, and } \therefore\ b = \frac{n - a^2}{2a} \text{ nearly,}$$

$$\text{whence, } \sqrt{n} = a + b = a + \frac{n - a^2}{2a} = \frac{a^2 + n}{2a} \text{ nearly:}$$

again, if this value be denoted by a', and b' be assumed such that

$$n = (a' + b')^2 = a'^2 + 2a'b' + b'^2:$$

$$\text{then, as before, we have } b' = \frac{n - a'^2}{2a'} \text{ nearly:}$$

$$\text{whence, } \sqrt{n} = a' + b' = a' + \frac{n - a'^2}{2a'} = \frac{a'^2 + n}{2a'} \text{ nearly,}$$

which is manifestly more nearly equal to the true value than the preceding one: and by a continuation of this process, we shall approximate more and more closely to the true value, till any required degree of accuracy is attained.

Ex. To approximate to the value of the square root of 3.

Here, we have $n = 3$, and $a = 1$:

$$\therefore\ \sqrt{3} = \frac{1 + 3}{2} = \frac{4}{2} = 2, \text{ the first approximation:}$$

again, we have $n = 3$, and $a' = 2$:

$$\therefore\ \sqrt{3} = \frac{4 + 3}{4} = \frac{7}{4} = 1.75, \text{ the second approximation:}$$

$$\text{also, we have } n=3, \text{ and } a''=\frac{7}{4}:$$

$$\therefore \sqrt{3}=\frac{97}{56}=1.732 \text{ \&c., the third approximation,}$$

which is accurate to the last place of decimals: and so on, to any degree of nicety.

115. When the cube root of a number consists of $2n+2$ figures, and $n+2$ figures have been found, the remaining n figures may be accurately determined by dividing by the corresponding *trial* divisor.

For, if the root be denoted by $a+b$, where a comprises $n+2$ figures with n ciphers annexed, and b contains n figures: then, when a is determined, the remainder is $3a^2b+3ab^2+b^3$, which, divided by the trial divisor $3a^2$, gives a quotient

$$=b+\frac{b^2}{a}+\frac{b^3}{3a^2}, \text{ differing from } b \text{ by } \frac{b^2}{a}+\frac{b^3}{3a^2}:$$

but b^2 must, in our system of notation, be less than $(10^n)^2$ or 10^{2n}, and a is not less than 10^{2n+1}:

$$\therefore \frac{b^2}{a} \text{ cannot be greater than } \frac{1}{10}:$$

also, b^3 must be less than $(10^n)^3$ or 10^{3n}, and $3a^2$ is not less than $3\times 10^{4n+2}$:

$$\therefore \frac{b^3}{3a^2} \text{ cannot be greater than } \frac{1}{10^{n+2}}:$$

whence it follows that the error cannot be greater than

$$\frac{1}{10}+\frac{1}{10^{n+2}}; \text{ or than } \frac{1}{10}+\frac{1}{100}=\frac{11}{100}, \text{ whatever } n \text{ may be:}$$

and therefore the last n figures will be accurately obtained by division only.

See the observation made at the conclusion of article (161), of the Arithmetic.

116. Fractional surds may always be transformed into others, by equal multiplication or division of their numerators and denominators.

Thus, $\dfrac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}}=\dfrac{a-\sqrt{a^2-x^2}}{x}$, by multiplying the numerator and denominator by the numerator:

also, $\dfrac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}}=\dfrac{x}{a+\sqrt{a^2-x^2}}$, by multiplying the numerator and denominator by the denominator.

117. Fractional surds may be reduced to their simplest forms, by dividing the numerators and denominators by their highest common factor.

Thus, $\dfrac{x^{\frac{3}{2}}+6x^2+5x^{\frac{5}{2}}}{1+x^{\frac{1}{2}}-x^{\frac{3}{2}}-x^2}$ is reduced to its simplest form $\dfrac{x^{\frac{3}{2}}+5x^2}{1-x^{\frac{3}{2}}}$, by dividing both the numerator and denominator by $1+x^{\frac{1}{2}}$, the greatest common measure, determined by the common rule.

118. Fractional surds may be reduced to others having the lowest common denominator, according to the principle of article (72).

Thus, if $\dfrac{1}{(1+x)^{\frac{1}{2}}}$, $\dfrac{2x}{(1+x)^{\frac{3}{2}}}$ and $\dfrac{3x^2}{(1+x)^{\frac{5}{2}}}$ be the fractions proposed, we shall have them expressed with a common denominator, by

$$\frac{(1+x)^2}{(1+x)^{\frac{5}{2}}},\ \frac{2x(1+x)}{(1+x)^{\frac{5}{2}}} \text{ and } \frac{3x^2}{(1+x)^{\frac{5}{2}}}.$$

119. A surd may be made to change its form by effecting an operation upon it, and then indicating its converse.

Thus, the square of $\sqrt{x} + \sqrt{1-x} = 1 + 2\sqrt{x - x^2}$: and therefore by the indication of the reverse operation,

$$\sqrt{x} + \sqrt{1-x} \text{ is equivalent to } \sqrt{1 + 2\sqrt{x - x^2}}.$$

120. An equality, whose members comprise surds, may always be divested of such quantities, by the performance of the proper involutions.

Thus, if $\sqrt{2ax + x^2 + \sqrt{a^2 + x^2}} = a - x$: we shall have

$$2ax + x^2 + \sqrt{a^2 + x^2} = a^2 - 2ax + x^2:$$

$$\therefore \sqrt{a^2 + x^2} = a^2 - 4ax:$$

whence, $a^2 + x^2 = (a^2 - 4ax)^2 = a^4 - 8a^3x + 16a^2x^2$, which does not involve a surd.

This operation is known by the name of *Clearing an Equation of Surds.*

IMAGINARY QUANTITIES.

121. DEF. From the circumstance that every algebraical symbol hitherto considered, whether it be affected with the sign + or −, when raised to an even power, gives a positive result, it follows that no even root of a negative quantity can be either positive or negative. The even roots of negative quantities, having therefore no symbolical representation in accordance with the views of Algebra, as far as we have yet considered it, can only be *indicated* or *expressed* by means of the radical sign, or corresponding fractional index; and hence arises a new species of symbolical expressions, known generally by the name of *Imaginary* or *Impossible* quantities.

Thus, the square root of $-a^2$ being neither $+a$ nor $-a$, is written $\sqrt{-a^2}$, and is equivalent to

$$\sqrt{a^2 \times (-1)} = \sqrt{a^2}\sqrt{-1} = \pm a\sqrt{-1},$$

which is said to be impossible or imaginary, in consequence of involving the symbol $\sqrt{-1}$, or $(-1)^{\frac{1}{2}}$.

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$$\sqrt{x}+\sqrt{1-x} \text{ is equivalent to } \sqrt{1+2\sqrt{x-x^2}}.$$

120. An equality, whose members comprise surds, may always be divested of such quantities, by the performance of the proper involutions.

Thus, if $\sqrt{2ax+x^2+\sqrt{a^2+x^2}} = a - x$: we shall have

$$2ax + x^2 + \sqrt{a^2+x^2} = a^2 - 2ax + x^2:$$

$$\therefore \sqrt{a^2+x^2} = a^2 - 4ax:$$

whence, $a^2 + x^2 = (a^2 - 4ax)^2 = a^4 - 8a^3x + 16a^2x^2$, which does not involve a surd.

This operation is known by the name of *Clearing an Equation of Surds.*

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$$\sqrt{a^2 \times (-1)} = \sqrt{a^2}\sqrt{-1} = \pm a\sqrt{-1},$$

which is said to be impossible or imaginary, in consequence of involving the symbol $\sqrt{-1}$, or $(-1)^{\frac{1}{2}}$.

(6) In Evolution:

$$\sqrt{a \pm b\sqrt{-1}} = \sqrt{\frac{a+\sqrt{a^2+b^2}}{2}} \pm \sqrt{\frac{a-\sqrt{a^2+b^2}}{2}}$$

$$= \sqrt{\frac{\sqrt{a^2+b^2}+a}{2}} \pm \sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}\sqrt{-1},$$

by article (107): and this, if a^2+b^2 be a complete square, will manifestly be of the specified form.

124. If $a+b\sqrt{-1}=c+d\sqrt{-1}$: then will $a=c$ and $b=d$.

For, if not, let $a=c+\alpha$, so that

$$c+\alpha+b\sqrt{-1}=c+d\sqrt{-1}, \text{ and } \therefore\ \alpha+b\sqrt{-1}=d\sqrt{-1}:$$

$$\text{then } \alpha=d\sqrt{-1}-b\sqrt{-1}=(d-b)\sqrt{-1}:$$

whence, squaring both members, we shall have

$$\alpha^2=-(d-b)^2, \text{ and } \therefore\ \alpha^2+(d-b)^2=0:$$

that is, a positive quantity is equal to a negative quantity, or the sum of two positive quantities is equal to 0, which is impossible: and therefore the equality cannot exist, unless $\alpha=0$, and $d-b=0$: whence, $a=c$ and $b=d$.

We will illustrate the use of this proposition in the following examples.

Ex. 1. Extract the square root of $5-12\sqrt{-1}$.

$$\text{Assume } \sqrt{5-12\sqrt{-1}}=\sqrt{u}-\sqrt{v}:$$

$\therefore\ 5-12\sqrt{-1}=u+v-2\sqrt{uv}$: from which we obtain

$$u+v=5, \text{ and } 2\sqrt{uv}=12\sqrt{-1}:$$

$$\begin{aligned} \therefore\ u^2+2uv+v^2 &= \ \ 25 \\ 4uv &= -144 \\ \hline \therefore\ u^2-2uv+v^2 &= \ \ 169 \end{aligned}$$

whence, $u - v = 13$; and this, with $u + v = 5$, gives

$2u = 18$, $u = 9$, $\sqrt{u} = 3$, and $2v = -8$, $v = -4$, $\sqrt{v} = 2\sqrt{-1}$:

$$\therefore \sqrt{5 - 12\sqrt{-1}} = 3 - 2\sqrt{-1}.$$

Ex. 2. Required the square root of $\dfrac{36\sqrt{-1} - 2}{2 + 3\sqrt{-1}}$.

Here,

$$\frac{36\sqrt{-1} - 2}{2 + 3\sqrt{-1}} = \frac{(36\sqrt{-1} - 2)(2 + 3\sqrt{-1})}{(2 + 3\sqrt{-1})(2 + 3\sqrt{-1})} = \frac{-112 + 66\sqrt{-1}}{(2 + 3\sqrt{-1})^2}:$$

$\therefore$ assuming $\sqrt{-112 + 66\sqrt{-1}} = \sqrt{u} + \sqrt{v}$, and proceeding as before, we shall find the square root required $= \dfrac{3 + 11\sqrt{-1}}{2 + 3\sqrt{-1}}$

$$= \frac{(3 + 11\sqrt{-1})(2 - 3\sqrt{-1})}{(2 + 3\sqrt{-1})(2 - 3\sqrt{-1})} = \frac{39 + 13\sqrt{-1}}{13} = 3 + \sqrt{-1}.$$

125. Whenever imaginary quantities are used as *indices*, the principle of the *Permanence of Equivalent Forms* leads to the adoption of the general *Theory of Indices* as laid down in the preceding pages.

Thus, $e^{\pm\alpha\sqrt{-1}} \times e^{\pm\beta\sqrt{-1}}$ is assumed to be equivalent to

$$e^{\pm(\alpha+\beta)\sqrt{-1}}: \; e^{\pm\alpha\sqrt{-1}} \div e^{\pm\beta\sqrt{-1}} \text{ to } e^{\pm(\alpha-\beta)\sqrt{-1}}:$$

$$\text{and } (e^{\pm\alpha\sqrt{-1}})^m \text{ to } e^{\pm m\alpha\sqrt{-1}},$$

where m may be integral or fractional, or indeed of any form whatever: and it is clear, from the nature of the case, that it would be futile to attempt to establish such equalities upon any other principle.

CHAPTER VI.

THE SOLUTION OF EQUATIONS.

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126. Def. An Equation implying the relation of *Equality* between its two members, is here considered as comprising *known* or *given* quantities represented by numbers and the letters a, b, c, &c., and *unknown* or *required* quantities denoted by the letters x, y, z, &c.: and the *Solution of Equations* is the expressing the values of the latter in terms of, or by means of, the former.

The Solutions of Equations will evidently be verified, if on substituting for x, y, z, &c. their values, both sides become *identical:* and these values, which are termed the *Roots* of the equations, are said to *fulfil* or *satisfy* the *conditions* which they involve.

Thus, in the equation $4x + 2 = 3x + 4$, the letter x denotes the unknown or required quantity, which is combined with the given numbers 2, 3 and 4; and the solution of this equation will be effected, if we can find such a numerical value of the symbol x, as will render identical, its two members $4x + 2$ and $3x + 4$.

A little consideration will, in this instance, shew that the value of x must be 2, as this manifestly gives $10 = 10$: that is, the number 2 renders the two members of the equation identical, and therefore satisfies the condition expressed by it: and it will appear upon *trial* that no other numerical magnitude can have the same effect.

127. In other instances, the root or roots might be found by trial; but when the terms of the equation are numerous, and the required quantity is much involved with those that

are known, it is evident that the mode above adopted, being regulated by no rule, would be quite incompetent to effect the solutions. On this account, recourse is had to a certain set of Principles for the Solution of Equations, which the following articles will explain and exemplify: and we will first suppose the equations under consideration to involve only one unknown quantity.

128. *Transposition.* When the unknown symbol appears in both members of an equation, it is evident from article (49), that the terms involving it may always be *transposed* in such a manner as to be found in the first member, whilst the known quantities make up the second.

Ex. 1. Let the equation be $4x - 2 = 3x + 4$: then, we shall have $4x - 3x = 4 + 2$, or $x = 6$.

Ex. 2. If the equation proposed be

$$ax^2 + bx - c = dx^2 - ex + f,$$

we shall have $ax^2 + bx - dx^2 + ex = c + f$:

and this may be written in the form,

$$(a - d)\, x^2 + (b + e)\, x = c + f,$$

which is arranged according to the dimensions of the unknown quantity x, the terms absolutely known constituting the second member of the result.

129. *Clearing of Fractions.* When any of the terms involving the unknown quantity, are of a fractional form, the equation may always be *divested* of fractions, by means of article (80).

Ex. 1. In the equation $\frac{x}{3} + 6 = \frac{2x}{5} - 3$: if we multiply all the terms by 15, the least common multiple of 3 and 5, we shall obtain,

$$5x + 90 = 6x - 45,$$

which does not involve fractional coefficients of the unknown symbol.

Ex. 2. If the equation be $\frac{ax}{b-x} = \frac{b}{a+x}$, we have

$$ax(a+x) = b(b-x), \text{ or } a^2x + ax^2 = b^2 - bx,$$

which will become arranged according to the dimensions of x, by tranposition: thus,

$$ax^2 + (a^2 + b)x = b^2.$$

130. *Clearing of Surds.* When the unknown quantity in an equation, appears in the form of a surd, the equation may always be made to assume a *rational* form, by proper transpositions and involutions of both its members, agreeably to the purport of article (120).

Ex. 1. Let the proposed equation be $x - 4 = 2 + \sqrt{x+3}$: then, by transposition, we have,

$$x - 6 = \sqrt{x+3}:$$

and squaring both sides, we obtain the rationalized equation,

$$x^2 - 12x + 36 = x + 3, \text{ or } x^2 - 13x = -33.$$

Ex. 2. If the equation be $a + \sqrt{2ax + x^2} = 2x$, we have

$$\sqrt{2ax + x^2} = 2x - a:$$

which, by involution of both the members, produces

$$2ax + x^2 = 4x^2 - 4ax + a^2, \text{ or } -3x^2 + 6ax = a^2:$$

and this, by changing the signs of all the terms, or multiplying them all by -1, becomes

$$3x^2 - 6ax = -a^2.$$

131. *Reducing to proper Forms.* By means of the last three articles, every equation however complicated may be reduced so as to involve only *integral* and *positive* powers of the unknown symbol: and if the coefficient of its highest power be not 1, the equal division of the terms of the equation by the coefficient, will always reduce it to one or other of the following forms:

(1) $x = p$:

(2) $x^2 - px = q$:

(3) $x^3 - px^2 + qx = r$: &c.

wherein the symbols p, q, r, &c. may be either positive or negative, integral, fractional or irrational.

The first is styled a *simple* equation, or equation of the *first order* or *degree:* the second a *quadratic* equation, or equation of the *second order:* the third a *cubic* equation, or equation of the *third order:* and so on.

At present our attention will be confined to the solution of such equations only as belong to the first two classes above enumerated, or may be reduced to them by substitutions or other artifices.

SIMPLE EQUATIONS.

132. Since every equation coming under this head is capable of reduction to the form,

$$x = p:$$

it is obvious that its solution is already effected, and that there is only one value of x which satisfies the proposed condition: so that every simple equation has *one*, and *only* one root.

This will fully appear in the treatment of the following equations: as also in the subsequent problems, whose solutions are dependent upon simple equations.

Ex. 1. Given $5x - 15 = 2x + 6$, to find x.

Here, $5x - 2x = 6 + 15$:

that is, $3x = 21$, and $\therefore\ x = \dfrac{21}{3} = 7$.

Ex. 2. Given $4(x - 3) + 3x + 1 = 2(x + 2)$, to find x.

Here, by effecting the multiplications indicated, we have

$$4x - 12 + 3x + 1 = 2x + 4:$$

whence, by transposition, we obtain

$$4x + 3x - 2x = 4 + 12 - 1:$$

$$\text{that is, } 5x = 15, \text{ and } \therefore\ x = \frac{15}{5} = 3.$$

Ex. 3. Given $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} - \frac{x}{5} = x - 7$, to find x.

Here, the least common multiple of 2, 3, 4 and 5 being 60, we multiply every term of the equation by 60, in order to clear it of fractions: and thus we have

$$30x + 20x + 15x - 12x = 60x - 420:$$

$\therefore$ by transposition and reduction, we obtain

$$-7x = -420, \text{ and } \therefore\ x = \frac{-420}{-7} = 60.$$

Ex. 4. Solve the equation $\frac{3x-1}{7} + \frac{11-4x}{3} = \frac{12}{7}$.

Here, multiplying every term by 21, we have

$$9x - 3 + 77 - 28x = 36:$$

$$\therefore\ 9x - 28x = 36 + 3 - 77, \text{ or } -19x = -38,$$

$$\text{which gives } x = \frac{-38}{-19} = 2.$$

Ex. 5. Given $\frac{1}{7}\left(x - \frac{1}{2}\right) - \frac{1}{5}\left(\frac{2}{3} - x\right) = 1\frac{13}{30}$, to find x.

When the indicated operations are effected, the equation is

$$\frac{x}{7} - \frac{1}{14} - \frac{2}{15} + \frac{x}{5} = \frac{43}{30}:$$

and multiplying every term by 210, the least common multiple of the denominators, we have

$$30x - 15 - 28 + 42x = 301:$$

$$\text{that is, } 72x = 344, \text{ and } \therefore\ x = \frac{344}{72} = \frac{43}{9} = 4\frac{7}{9}.$$

Ex. 6. From the equation $\dfrac{49}{8x+1} = \dfrac{70}{12x+1}$, find x.

By dividing both members by 7, we have

$$\frac{7}{8x+1} = \frac{10}{12x+1}:$$

$$\therefore\ 84x + 7 = 80x + 10,\ 4x = 3, \text{ and } x = \frac{3}{4}.$$

Ex. 7. Solve the equation $\dfrac{a}{1-bx} = \dfrac{b}{1-ax}$.

This equation is reduced by the inversion of its terms, to

$$\frac{1-bx}{a} = \frac{1-ax}{b}, \text{ or } \frac{1}{a} - \frac{b}{a}x = \frac{1}{b} - \frac{a}{b}x:$$

$$\therefore\ \left(\frac{a}{b} - \frac{b}{a}\right)x = \frac{1}{b} - \frac{1}{a}, \text{ or } \frac{a^2-b^2}{ab}x = \frac{a-b}{ab}:$$

$$\text{that is, } x = \frac{a-b}{a^2-b^2} = \frac{1}{a+b}.$$

Ex. 8. Given $\sqrt{x} + \sqrt{x+3} = \dfrac{12}{\sqrt{x+3}}$, to find x.

Multiplying every term by $\sqrt{x+3}$, we have

$$\sqrt{x^2+3x} + x + 3 = 12, \text{ or } \sqrt{x^2+3x} = 9 - x:$$

and squaring both members, we obtain

$$x^2 + 3x = 81 - 18x + x^2:$$

$$\text{whence, } 21x = 81, \text{ and } \therefore\ x = \frac{81}{21} = \frac{27}{7} = 3\frac{6}{7}.$$

Ex. 9. Given $\sqrt[m]{x+2} = \sqrt[2m]{x^2+10x+1}$, to fin

This equation may be written in the form,

$$(x+2)^{\frac{1}{m}} = (x^2 \quad\quad)^{\frac{1}{n}}:$$

whence, raising each member to the power denoted by $2m$,

$$\text{we have } (x+2)^2 = x^2 + 10x + 1:$$

$$\text{that is, } x^2 + 4x + 4 = x^2 + 10x + 1:$$

$$\text{whence we find } 6x = 3, \text{ and } \therefore\ x = \frac{1}{2}.$$

Ex. 10. Given $\{x + (4ax + 17a^2)^{\frac{1}{2}}\}^{\frac{1}{2}} = x^{\frac{1}{2}} + a^{\frac{1}{2}}$, to find x.

Squaring both members of the equation, we obtain

$$x + (4ax + 17a^2)^{\frac{1}{2}} = x + 2a^{\frac{1}{2}}x^{\frac{1}{2}} + a:$$

$$\text{whence we find, } (4ax + 17a^2)^{\frac{1}{2}} = 2a^{\frac{1}{2}}x^{\frac{1}{2}} + a:$$

and by squaring again, we have $4ax + 17a^2 = 4ax + 4a^{\frac{3}{2}}x^{\frac{1}{2}} + a^2$:

$$\text{which gives } 4a^{\frac{3}{2}}x^{\frac{1}{2}} = 16a^2,\ x^{\frac{1}{2}} = 4a^{\frac{1}{2}}, \text{ and } x = 16a.$$

Ex. 11. Given $\sqrt{x+5} \times \sqrt{x+12} = 8 + x$, to find x.

Squaring both members of the equation, we have

$$(x+5) \times (x+12), \text{ or } x^2 + 17x + 60 = 64 + 16x + x^2:$$

$$\text{whence, } 17x - 16x = 64 - 60, \text{ or } x = 4.$$

Ex. 12. Given $\dfrac{x-1}{\sqrt{x}+1} = 1 + \dfrac{\sqrt{x}-1}{2}$, to find x.

Since, by actual division, $\dfrac{x-1}{\sqrt{x}+1} = \sqrt{x} - 1$: we have

$$\sqrt{x} - 1 = 1 + \frac{\sqrt{x}-1}{2}, \text{ or } 2\sqrt{x} - 2 = 2 + \sqrt{x} - 1:$$

$$\text{whence, by transposition \&c., } \sqrt{x} = 3, \text{ and } \therefore\ x = 9.$$

Ex. 13. Given $\dfrac{\sqrt{ax}+\sqrt{b}}{\sqrt{ax}-\sqrt{b}} = \dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{b}}$, to find x.

Clearing the equation of the fractional forms, we have

$$\sqrt{abx} + b = a\sqrt{x} + \sqrt{abx} - \sqrt{ab} - b:$$

$$\therefore\ 2b+\sqrt{ab}=a\sqrt{x},\ \text{ or }\ \sqrt{x}=\frac{\sqrt{b}\,(a+2\sqrt{b})}{a}:$$

whence, by involution, we find $x=\dfrac{b(a+2\sqrt{b})^2}{a^2}$.

Ex. 14. Given

$$\sqrt{\sqrt{x}+\sqrt{5}}+\sqrt{\sqrt{x}-\sqrt{5}}=\sqrt{2\sqrt{x}+4},\ \text{to find } x.$$

Squaring both members of the equation, we have

$$\sqrt{x}+\sqrt{5}+2\sqrt{x-5}+\sqrt{x}-\sqrt{5}=2\sqrt{x}+4,\ \text{ or }\ 2\sqrt{x-5}=4:$$

whence, by repeating the same operation, we obtain

$$4x-20=16:\ \text{and}\ \therefore\ 4x=36,\ \text{or}\ x=9.$$

133. When the terms of an equation contain both *simple* and *compound* denominators, it will generally be found convenient to divest it of the simple denominators at first, and afterwards of those which are compound.

Ex. Given $\dfrac{6x+7}{9}+\dfrac{7x+13}{6x+3}=\dfrac{2x+4}{3}$, to find x.

Here, multiplying every term by 9, we have

$$6x+7+\frac{21x+39}{2x+1}=6x+12:$$

which gives $\dfrac{21x+39}{2x+1}=5$: $\therefore\ 21x+39=10x+5$:

whence, $11x=-34$, and $\therefore\ x=-\dfrac{34}{11}=-3\dfrac{1}{11}$.

134. Whenever it is found that one or more of the terms of an equation are in the form of *complex fractions*, it will be desirable to remove this form, by equal multiplication of the numerators and denominators.

Ex. Given $\dfrac{2x+8\frac{1}{2}}{9}-\dfrac{13x-2}{17x-32}+\dfrac{x}{3}=\dfrac{7x}{12}-\dfrac{x+16}{36}$, to

whence, raising each member to the power denoted by $2m$,

$$\text{we have } (x+2)^2 = x^2 + 10x + 1:$$

$$\text{that is, } x^2 + 4x + 4 = x^2 + 10x + 1:$$

$$\text{whence we find } 6x = 3, \text{ and } \therefore\ x = \frac{1}{2}.$$

Ex. 10. Given $\{x + (4ax + 17a^2)^{\frac{1}{2}}\}^{\frac{1}{2}} = x^{\frac{1}{2}} + a^{\frac{1}{2}}$, to find x.

Squaring both members of the equation, we obtain

$$x + (4ax + 17a^2)^{\frac{1}{2}} = x + 2a^{\frac{1}{2}}x^{\frac{1}{2}} + a:$$

$$\text{whence we find, } (4ax + 17a^2)^{\frac{1}{2}} = 2a^{\frac{1}{2}}x^{\frac{1}{2}} + a:$$

and by squaring again, we have $4ax + 17a^2 = 4ax + 4a^{\frac{3}{2}}x^{\frac{1}{2}} + a^2$:

which gives $4a^{\frac{3}{2}}x^{\frac{1}{2}} = 16a^2$, $x^{\frac{1}{2}} = 4a^{\frac{1}{2}}$, and $x = 16a$.

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Squaring both members of the equation, we have

$$(x+5) \times (x+12), \text{ or } x^2 + 17x + 60 = 64 + 16x + x^2:$$

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Ex. 12. Given $\dfrac{x-1}{\sqrt{x}+1} = 1 + \dfrac{\sqrt{x}-1}{2}$, to find x.

Since, by actual division, $\dfrac{x-1}{\sqrt{x}+1} = \sqrt{x} - 1$: we have

$$\sqrt{x} - 1 = 1 + \frac{\sqrt{x}-1}{2}, \text{ or } 2\sqrt{x} - 2 = 2 + \sqrt{x} - 1:$$

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$$\sqrt{abx} + b = a\sqrt{x} + \sqrt{abx} - \sqrt{ab} - b:$$

$$\therefore\ 2b + \sqrt{ab} = a\sqrt{x}, \text{ or } \sqrt{x} = \frac{\sqrt{b}(a + 2\sqrt{b})}{a}:$$

whence, by involution, we find $x = \dfrac{b(a + 2\sqrt{b})^2}{a^2}$.

Ex. 14. Given

$$\sqrt{\sqrt{x} + \sqrt{5}} + \sqrt{\sqrt{x} - \sqrt{5}} = \sqrt{2\sqrt{x} + 4}, \text{ to find } x.$$

Squaring both members of the equation, we have

$$\sqrt{x} + \sqrt{5} + 2\sqrt{x - 5} + \sqrt{x} - \sqrt{5} = 2\sqrt{x} + 4, \text{ or } 2\sqrt{x - 5} = 4:$$

whence, by repeating the same operation, we obtain

$$4x - 20 = 16: \text{ and } \therefore\ 4x = 36, \text{ or } x = 9.$$

133. When the terms of an equation contain both *simple* and *compound* denominators, it will generally be found convenient to divest it of the simple denominators at first, and afterwards of those which are compound.

Ex. Given $\dfrac{6x + 7}{9} + \dfrac{7x + 13}{6x + 3} = \dfrac{2x + 4}{3}$, to find x.

Here, multiplying every term by 9, we have

$$6x + 7 + \frac{21x + 39}{2x + 1} = 6x + 12:$$

which gives $\dfrac{21x + 39}{2x + 1} = 5$: $\therefore\ 21x + 39 = 10x + 5$:

whence, $11x = -34$, and $\therefore\ x = -\dfrac{34}{11} = -3\dfrac{1}{11}$.

134. Whenever it is found that one or more of the terms of an equation are in the form of *complex fractions*, it will be desirable to remove this form, by equal multiplication of the numerators and denominators.

Ex. Given $\dfrac{2x + 8\frac{1}{2}}{9} - \dfrac{13x - 2}{17x - 32} + \dfrac{x}{3} = \dfrac{7x}{12} - \dfrac{x + 16}{36}$, to find x.

As no general rules, applicable to the infinite variety of problems that may occur, can be laid down, the student must be content with such directions upon the subject as he may be enabled to collect from particular examples.

Ex. 1. Find a number whose fourth part exceeds its fifth part by 2.

Let x be taken to represent the required number: then $\frac{x}{4}$ and $\frac{x}{5}$ will be adequate representations of its fourth and fifth parts: and the condition being that the former shall exceed the latter by 2, we shall have

$$\frac{x}{4} - \frac{x}{5} = 2:$$

which is the algebraical expression of the proposed problem: to solve this equation, we have $5x - 4x = 40$, and $\therefore x = 40$: that is, the required number is 40, as is easily verified:

$$\text{for, } \frac{40}{4} - \frac{40}{5} = 10 - 8 = 2.$$

Ex. 2. Divide 21 into two parts, so that ten times one of them may exceed nine times the other by 1.

If x represent one of the parts, $21 - x$ will evidently represent the other: whence, agreeably to the question, we must have

$$10x - 9(21 - x) = 1, \text{ or } 10x - 189 + 9x = 1,$$

which is the translation of the problem into algebraical language, exhibited by means of an equation: this solved, gives

$$19x = 190, \text{ and } \therefore x = 10:$$

so that the parts sought, are

$$x = 10, \text{ and } 21 - x = 21 - 10 = 11:$$

and it is seen immediately that 10 and 11 fulfil the specified condition.

Ex. 3. Find two numbers whose sum shall be 10, and quotient 3.

Taking x to represent the less of the two numbers, we shall have the greater $= 10 - x$: whence, the second condition gives the equation

$$\frac{10 - x}{x} = 3:$$

$$\therefore\ 10 - x = 3x,\ 4x = 10, \text{ and } x = 2\tfrac{1}{2}, \text{ the less:}$$

$$\text{whence, } 10 - x = 10 - 2\tfrac{1}{2} = 7\tfrac{1}{2}, \text{ the greater:}$$

and from this it appears that there do not exist two *whole numbers* possessing the properties enunciated, but that they both belong to the *fractional* magnitudes $2\frac{1}{2}$ and $7\frac{1}{2}$.

Ex. 4. A father's age is twice as great as that of his son, but 10 years hence, it will be three times as great: find the age of each.

Let x denote the son's age, then $2x$ will represent that of the father: also, at the end of 10 years, $x + 10$ and $2x + 10$ will denote the ages of the son and father respectively: and the condition of the question gives the equation

$$2x + 10 = 3(x + 10), \text{ or } 2x + 10 = 3x + 30:$$

from which we find immediately, $x = -20$.

As negative quantities have no existence in *Arithmetical Algebra*, this result proves that the conditions of the proposed problem are absurd, as also evidently appears from the nature of the case: but still the negative value above found will admit of an interpretation, agreeably to the principles of *Symbolical Algebra*, as follows.

If in the equation $2x + 10 = 3x + 30$, which expresses algebraically the condition implied in the question, we substitute the negative symbol $-x$ for x, we shall have

$$-2x + 10 = -3x + 30, \text{ or } 2x - 10 = 3(x - 10):$$

which is manifestly the algebraical expression of the following problem:

"A father's age is twice as great as that of his son, but 10 years ago, it was three times as great: find the age of each."

This differs from the original problem only in making use of the word *ago*, instead of the word *hence;* and from the two considered in connexion with each other, we are led to conclude that if *prospective* time from any epoch be considered positive, *retrospective* time reckoned from the same epoch, must be regarded as negative; and *vice versâ.*

Similar explanations of negative results may be given in most other cases of this description.

Ex. 5. A gentleman wishing to relieve a number of beggars, finds that if he give them 6*d.* a-piece, he will have 20*d.* left: and that he has not enough by 14*d.* to give them 8*d.* a-piece: required the number of beggars, and the money he possesses.

Let $x =$ the number of beggars: then, according to the enunciation of the question, both $6x + 20$ and $8x - 14$ will be adequate representations of his money in pence: whence

$6x + 20 = 8x - 14$: and $\therefore x = 17$, the number of beggars:

and the sum of money he has $= 6x + 20 = 102 + 20 = 122d.$

Ex. 6. Three persons A, B, C are possessed of certain sums of money, such that A and B together have £120: A and C together have £140: and B and C together have £150: what is the sum possessed by each?

Let $x = A$'s money: then, from the nature of the case,

$$120 - x = B\text{'s money, and } 140 - x = C\text{'s money}:$$

$$\therefore B \text{ and } C \text{ together have } 120 - x + 140 - x = 260 - 2x:$$

$$\text{whence, } 260 - 2x = 150, \text{ by the question}:$$

$$\text{and this gives } x = £55 = A\text{'s money}:$$

$$\therefore 120 - x = £65 = B\text{'s money}:$$

$$\text{and } 140 - x = £85 = C\text{'s money}.$$

Ex. 7. *A* and *B* play together for a stake of 12*s.*: if *A* win, he will have thrice as much money as *B*; but if he lose, he will have only twice as much: how much does each possess at first?

Let x denote *A*'s money at first in shillings:

then, $\dfrac{x+12}{3}$ = *B*'s money after losing 12*s.* to *A*:

$\therefore\ \dfrac{x+12}{3}+12=\dfrac{x+48}{3}$ = *B*'s money at first:

also, $x-12$ = *A*'s money after losing 12*s.*:

and $\dfrac{x+48}{3}+12=\dfrac{x+84}{3}$ = *B*'s money after winning 12*s.*:

whence, $x-12=2\left(\dfrac{x+84}{3}\right)$, or $3x-36=2x+168$:

$\therefore\ x=168+36=204s.$ = *A*'s money at first:

and $\dfrac{x+48}{3}=\dfrac{204+48}{3}=\dfrac{252}{3}=84s.$ = *B*'s money at first.

Ex. 8. An egg-merchant meeting with three customers, sells to the first of them half of his stock and one egg more: to the second he disposes of half the remainder and two eggs more: and to the third half of what he then had left and three eggs more, when he finds that he has parted with his whole stock: what number of eggs had he at first?

Let x = the number he had at first:

$\therefore\ \dfrac{x}{2}+1$ = the number sold to the first customer:

and $\therefore\ \dfrac{x}{2}-1$ = the number then left:

also, $\dfrac{x}{4}-\dfrac{1}{2}+2$ = the number sold to the second customer:

$$\text{and} \quad \therefore \ \frac{x}{4} - \frac{1}{2} - 2 = \text{the number then remaning:}$$

$$\text{again,} \ \frac{x}{8} - \frac{1}{4} - 1 + 3 = \text{the number sold to the third customer:}$$

$$\text{and} \quad \therefore \ \frac{x}{8} - \frac{1}{4} - 1 - 3 = 0, \text{ by the question:}$$

whence, $x = 34$, the required number.

Ex. 9. A person disposes of turkeys at as many shillings each as the number he has, and returning back 1*s*. finds that if he had had one more to sell on the same condition, and had returned back 2*s*., he would have received 20*s*. more by the transaction: what number did he dispose of?

Let x denote the required number:

$\therefore \ x^2 - 1 =$ the number of shillings received:

also, $(x+1)^2 - 2 = x^2 + 2x - 1 =$ the number of shillings he would have received on the second hypothesis: whence, by the question, we have

$$x^2 + 2x - 1 = x^2 - 1 + 20, \text{ or } x = 10;$$

that is, 10 is the number he disposed of, and the result is very easily verified.

Ex. 10. If A and B can do a piece of work in m days; A and C in n days: and B can do p times as much as C in a given time: find the time in which B and C can do it.

Let a denote the proposed work:

$$\therefore \ \frac{a}{m} = \text{the work done by } A \text{ and } B \text{ in one day:}$$

$$\frac{a}{n} = \text{the work done by } A \text{ and } C \text{ in one day:}$$

let x represent the time in which C alone can do it:

$$\therefore \ \frac{a}{x} = \text{the work done by } C \text{ in one day:}$$

and $\dfrac{pa}{x}$ = the work done by B in one day:

whence, $\dfrac{(p+1)a}{x}$ = the work done by B and C in one day:

and the required number of days $= a \div \dfrac{(p+1)a}{x} = \dfrac{x}{p+1}$,

which is therefore independent of the whole work done:

also, $\dfrac{a}{m} - \dfrac{pa}{x} + \dfrac{a}{x}$ = work done by A and C in one day $= \dfrac{a}{n}$;

which gives immediately, $x = (p-1)\dfrac{mn}{n-m}$:

$$\therefore \text{ the required time } = \frac{x}{p+1} = \left(\frac{p-1}{p+1}\right)\left(\frac{mn}{n-m}\right).$$

If p be greater than 1, this result will be accordant with the views of arithmetic, provided $\dfrac{mn}{n-m}$ be positive, or n be greater than m: and that this ought to be the case, will appear from considering that if B can do *more* work than C in a given time, A and B can do *more* than A and C in a given time, and consequently that m is less than n: and *vice versâ*.

If $p = 1$, we shall manifestly have $m = n$, by the same mode of reasoning: and the required time is expressed by $\dfrac{0}{0}$, which being *indeterminate*, or capable of *any value whatever*, shews that the conditions of the question are in this case insufficient for ensuring its solution.

If p be less than 1, and n be greater than m, the value of x will be $-(1-p)\dfrac{mn}{n-m}$:

$\therefore$ the work done by C in one day $= \dfrac{a}{x} = -\dfrac{a}{1-p}\left(\dfrac{n-m}{mn}\right)$:

and the work done by B and C in one day

$$=\frac{(p+1)a}{x}=-\frac{(1+p)a}{1-p}\left(\frac{n-m}{mn}\right):$$

and the latter being, by article (78), less than the former, proves that the effect of B's exertions is to *impede*, and not to *promote* the completion of the work.

QUADRATIC EQUATIONS.

138. DEF. The equations belonging to this class, being reducible to one or other of the forms,

$$x^2=p, \text{ or } x^2-px=q:$$

it remains to devise general methods by which the values of x may be determined from each, the former being styled a *Pure Quadratic*, and the latter an *Adfected Quadratic* Equation.

PURE QUADRATIC EQUATIONS.

139. From the equation $x^2=p$, we find the values of x immediately, by the extraction of the square roots of both members, which gives $x=\pm\sqrt{p}$: that is, the values of x are $+\sqrt{p}$ and $-\sqrt{p}$: and it is very readily seen that they both satisfy the condition.

Hence, every pure quadratic equation properly so called, has *two*, and *only* two roots, which are equal in *magnitude*, but differ in their *algebraical signs*.

Also, if a and $-a$ be taken to denote these two roots, we have $-a^2=-p$; and thus the equation is equivalent to

$$x^2=a^2, \text{ or } x^2-a^2=0, \text{ or } (x-a)(x+a)=0.$$

Ex. 1. Given $51x^2-96=39x^2+96$, to find x.

Here, $51x^2-39x^2=96+96$, or $12x^2=192$:

whence, $x^2=16$, and $\therefore x=\pm 4$:

that is, the values of x are 4 and -4; the former giving an *arithmetical*, and the latter a *symbolical* solution of the equation.

Ex. 2. Given $\sqrt{x+a} = \sqrt{x + \sqrt{b^2 + x^2}}$, to find x.

Here, we have $x + a = x + \sqrt{b^2 + x^2}$, or $a = \sqrt{b^2 + x^2}$:

whence, $a^2 = b^2 + x^2$, $x^2 = a^2 - b^2$, and $\therefore\ x = \pm\sqrt{a^2 - b^2}$:

or, the two values of x are $+\sqrt{a^2 - b^2}$ and $-\sqrt{a^2 - b^2}$,

which become identical, if the signs be removed.

140. All other equations reducible to the form $x^m = p$, may evidently be solved by the same method: thus, $x = \sqrt[m]{p}$, the double sign $\pm$ being used, when m is an even number.

Ex. Given $a = \sqrt[3]{x^3 + \sqrt{x^6 - a^6}}$, to find x.

Here, $a^3 = x^3 + \sqrt{x^6 - a^6}$: $\therefore\ a^3 - x^3 = \sqrt{x^6 - a^6}$:

whence, $a^6 - 2a^3x^3 + x^6 = x^6 - a^6$, or $2a^3x^3 = 2a^6$:

$\therefore\ x^3 = a^3$, and $x = a$, by evolution.

ADFECTED QUADRATIC EQUATIONS.

141. *To investigate the solution of an adfected quadratic equation, by completing the square.*

Let $x^2 - px = q$, be the equation reduced to its proper form, and suppose r to be the quantity which, added to both its members, shall render the former a complete square, so that $x^2 - px + r = q + r$: then, extracting the square root by the ordinary process,

$$\begin{array}{rl}
 & x^2 - px + r\ (x - \tfrac{1}{2}p \\
 & x^2 \\
\hline
2x - \tfrac{1}{2}p) & -px + r \\
 & -px + \tfrac{1}{4}p^2 \\
\hline
 & r - \tfrac{1}{4}p^2 \\
\hline
\end{array}$$

we must manifestly have $r - \frac{1}{4}p^2 = 0$,

which gives $r = \frac{1}{4}p^2 = (-\frac{1}{2}p)^2$:

that is, in an adfected quadratic equation reduced to its proper form, the quantity to be added to both members to render the former a complete square, is the *square of half the coefficient* of the *second term:*

$$\text{whence, } x^2 - px + \tfrac{1}{4}p^2 = \tfrac{1}{4}p^2 + q:$$

$$\therefore\ x - \tfrac{1}{2}p = \pm\sqrt{\tfrac{1}{4}p^2 + q} = \pm\tfrac{1}{2}\sqrt{p^2 + 4q};$$

$$\text{and } x = \tfrac{1}{2}(p \pm \sqrt{p^2 + 4q}).$$

0 142. COR. 1. In the preceding article, we have found the values of x to be

$$\tfrac{1}{2}(p + \sqrt{p^2 + 4q}) \text{ and } \tfrac{1}{2}(p - \sqrt{p^2 + 4q}):$$

from which it appears that every quadratic equation properly so called, has *two*, and *only* two roots.

143. COR. 2. We may hence express any quadratic equation, as $x^2 - px + q = 0$, in terms of its roots.

For, proceeding as before, we find the values of x to be

$$\tfrac{1}{2}(p + \sqrt{p^2 - 4q}) \text{ and } \tfrac{1}{2}(p - \sqrt{p^2 - 4q}):$$

and denoting these roots by α, β respectively, we have

$$\alpha + \beta = \tfrac{1}{2}(p + \sqrt{p^2 - 4q}) + \tfrac{1}{2}(p - \sqrt{p^2 - 4q})$$
$$= \tfrac{1}{2}p + \tfrac{1}{2}p = p:$$

$$\alpha\beta = \tfrac{1}{2}(p + \sqrt{p^2 - 4q}) \times \tfrac{1}{2}(p - \sqrt{p^2 - 4q})$$
$$= \tfrac{1}{4}\{p^2 - (p^2 - 4q)\} = \tfrac{1}{4}(4q) = q:$$

whence, the equation $x^2 - px + q = 0$, is manifestly equivalent to

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0:$$

where the coefficient of the second term, with its sign changed, is equal to the sum of the roots, and the third term is equal to their product.

This conclusion will be found to be of great utility in the various applications of Algebra to other subjects.

144. Cor. 3. Since $x^2 - (\alpha + \beta)x + \alpha\beta = 0$, is equivalent to $(x - \alpha)(x - \beta) = 0$, the roots of the equation will at once be manifest, whenever it can be expressed in the latter form, by making successively,

$$x - \alpha = 0, \text{ or } x = \alpha, \text{ and } x - \beta = 0, \text{ or } x = \beta.$$

145. Cor. 4. If the equation be $x^2 - px + q = 0$, and $p^2 = 4q$, we have $\alpha = \frac{1}{2}p$, and $\beta = \frac{1}{2}p$: and therefore the two roots are *equal:* if $p^2 > 4q$, the values of α and β are both positive *real* magnitudes: but, if $p^2 < 4q$, the values of α and β are both *imaginary* quantities, and the equation is incapable of an arithmetical solution.

If the equation be $x^2 - px - q = 0$, the roots are

$$\tfrac{1}{2}(p + \sqrt{p^2 + 4q}) \text{ and } \tfrac{1}{2}(p - \sqrt{p^2 + 4q}),$$

which are both *real* magnitudes, the former being positive and the latter negative, whatever numerical relation subsist between the values of p and q.

146. Cor. 5. If $p^2 < 4q$, the trinomial $x^2 - px + q$ can never become negative, whatever real value be assigned to x: for,

$$\begin{aligned} x^2 - px + q &= x^2 - px + \tfrac{1}{4}p^2 + \tfrac{1}{4}(4q - p^2) \\ &= (x - \tfrac{1}{2}p)^2 + \tfrac{1}{4}(4q - p^2), \end{aligned}$$

and both these quantities are essentially positive, independently of the value of x.

147. Cor. 6. The trinomial $x^2 - px + q$, may be resolved into its simple factors, by finding the roots α, β of the equation $x^2 - px + q = 0$: for then,

$$x^2 - px + q = (x - \alpha)(x - \beta), \text{ by article (144)},$$

where any value whatever may be given to x, because the members of this equality are identical.

Ex. 1. Given $x^2 - 6x + 12 = 4$, to find x.

First, we have $x^2 - 6x = -8$:

then, completing the square by adding to both members 9, the square of half the coefficient of the second term, we obtain,

$$x^2 - 6x + 9 = 1:$$

whence, extracting the square roots of both sides, we find $x - 3 = \pm 1$, two simple equations:

$$\text{and} \quad \therefore \; x = \pm 1 + 3 = 4 \text{ and } 2.$$

Hence, also, $x^2 - 6x + 8 = 0$, having the roots 4 and 2, it follows, by the last article, that

$$x^2 - 6x + 8 = (x - 4)(x - 2).$$

Ex. 2. Given $x^2 + 7x = 8$, to find x.

$$\text{Here, } x^2 + 7x + \frac{49}{4} = 8 + \frac{49}{4} = \frac{32 + 49}{4}$$

$$= \frac{81}{4}:$$

$$\therefore \; x + \frac{7}{2} = \pm \frac{9}{2}, \text{ by extracting the square roots:}$$

$$\text{whence, } x = \frac{9}{2} - \frac{7}{2} = \frac{2}{2} = 1, \text{ or } = -\frac{9}{2} - \frac{7}{2} = -\frac{16}{2} = -8.$$

As before, we shall have $x^2 + 7x - 8 = (x - 1)(x + 8)$.

Ex. 3. Given $\dfrac{7}{4} - \dfrac{2x - 5}{x + 5} = \dfrac{3x - 7}{2x}$, to find x.

Here, multiplying every term by $4x$, we have

$$7x - \frac{8x^2 - 20x}{x + 5} = 6x - 14:$$

$$\text{whence, } x + 14 = \frac{8x^2 - 20x}{x + 5}, \text{ or } (x + 14)(x + 5) = 8x^2 - 20x:$$

$$\text{that is, } x^2 + 19x + 70 = 8x^2 - 20x:$$

$$\therefore \; 7x^2 - 39x = 70, \text{ and } x^2 - \frac{39}{7}x = 10:$$

$$\therefore\ x^2 - \frac{39}{7}x + \left(\frac{39}{14}\right)^2 = 10 + \frac{1521}{196} = \frac{1960 + 1521}{196}$$

$$= \frac{3481}{196}:$$

whence, $x - \frac{39}{14} = \pm \frac{59}{14}$, by extracting the square roots:

$$\therefore\ x = \frac{59}{14} + \frac{39}{14} = \frac{98}{14} = 7, \text{ or } x = -\frac{59}{14} + \frac{39}{14} = -\frac{20}{14} = -1\frac{3}{7}:$$

that is, the values of x are 7 and $-1\frac{3}{7}$:

$$\text{and } \therefore\ x^2 - \frac{39}{7}x - 10 = (x - 7)\left(x + \frac{10}{7}\right).$$

Ex. 4. Given $acx^2 - bcx + adx = bd$, to find x.

Here, dividing every term by ac, we have

$$x^2 - \left(\frac{b}{a} - \frac{d}{c}\right)x = \frac{bd}{ac}:$$

$$\therefore\ x^2 - \left(\frac{b}{a} - \frac{d}{c}\right)x + \frac{1}{4}\left(\frac{b}{a} - \frac{d}{c}\right)^2 = \frac{bd}{ac} + \frac{1}{4}\left(\frac{b}{a} - \frac{d}{c}\right)^2$$

$$= \frac{4bd}{4ac} + \frac{1}{4}\left(\frac{b^2}{a^2} - \frac{2bd}{ac} + \frac{d^2}{c^2}\right)$$

$$= \frac{1}{4}\left(\frac{b^2}{a^2} + \frac{2bd}{ac} + \frac{d^2}{c^2}\right) = \frac{1}{4}\left(\frac{b}{a} + \frac{d}{c}\right)^2:$$

$$\text{whence, } x - \frac{1}{2}\left(\frac{b}{a} - \frac{d}{c}\right) = \pm \frac{1}{2}\left(\frac{b}{a} + \frac{d}{c}\right):$$

$$\therefore\ x = \frac{1}{2}\left(\frac{b}{a} - \frac{d}{c}\right) \pm \frac{1}{2}\left(\frac{b}{a} + \frac{d}{c}\right):$$

$$\text{that is, } x = \frac{1}{2}\left(\frac{b}{a} - \frac{d}{c} + \frac{b}{a} + \frac{d}{c}\right) = \frac{1}{2}\frac{2b}{a} = \frac{b}{a}:$$

$$\text{and } x = \frac{1}{2}\left(\frac{b}{a} - \frac{d}{c} - \frac{b}{a} - \frac{d}{c}\right) = -\frac{1}{2}\frac{2d}{c} = -\frac{d}{c}.$$

As in the preceding examples, we have here

$$x^2 - \left(\frac{b}{a} - \frac{d}{c}\right) x - \frac{bd}{ac} = \left(x - \frac{b}{a}\right)\left(x + \frac{d}{c}\right).$$

Ex. 5. Given $\sqrt{\frac{3x}{x+1}} - \sqrt{\frac{x+1}{3x}} = 2$, to find x.

Squaring both sides, we have $\frac{3x}{x+1} - 2 + \frac{x+1}{3x} = 4$:

whence, we easily obtain $x^2 + 2x = \frac{1}{8}$:

$$\therefore\ x^2 + 2x + 1 = \frac{9}{8}, \text{ and } x + 1 = \pm\frac{3}{2\sqrt{2}}:$$

from which, $x = -1 + \frac{3}{2\sqrt{2}}$, and $x = -1 - \frac{3}{2\sqrt{2}}$, two irrational roots.

148. If an equation after the requisite reductions, assume the form,

$$x^{2m} - px^m = q:$$

where m is either positive or negative, integral or fractional, the solution may be effected by completing the square: thus,

$$x^{2m} - px^m + \tfrac{1}{4}p^2 = \tfrac{1}{4}(p^2 + 4q):$$

$$\therefore\ x^m - \tfrac{1}{2}p = \pm\tfrac{1}{2}\sqrt{p^2 + 4q}, \text{ and } x^m = \tfrac{1}{2}(p \pm \sqrt{p^2 + 4q}):$$

whence, $x = \left\{\tfrac{1}{2}(p \pm \sqrt{p^2 + 4q})\right\}^{\frac{1}{m}}$, comprises the required roots.

Ex. 1. Given $x^6 - 6x^3 = 16$, to find x.

Here, $x^6 - 6x^3 + 9 = 16 + 9 = 25$:

$$\therefore\ x^3 - 3 = \pm 5, \text{ and } x^3 = 8 \text{ or } -2:$$

$$\text{whence, } x = 8^{\frac{1}{3}} = 2, \text{ or } = (-2)^{\frac{1}{3}} = -\sqrt[3]{2}.$$

Ex. 2. Given $x^{-4} - 9x^{-2} + 20 = 0$, to find x.

Here, $x^{-4} - 9x^{-2} + \frac{81}{4} = \frac{81}{4} - 20 = \frac{1}{4}$:

$$\therefore\ x^{-2} - \frac{9}{2} = \pm\frac{1}{2}, \text{ and } x^{-2} = \pm\frac{1}{2} + \frac{9}{2} = 5 \text{ or } 4:$$

$$\text{that is, } \frac{1}{x^2} = 5 \text{ or } 4, \text{ and } \therefore\ x^2 = \frac{1}{5} \text{ or } \frac{1}{4}:$$

$$\text{whence, } x = \pm\frac{1}{\sqrt{5}}, \text{ and } x = \pm\frac{1}{2}:$$

and the equation contains two rational, and two irrational roots.

Ex. 3. Given $\sqrt[4]{x} + 7\sqrt{x} = 116$, to find x.

Here, $7x^{\frac{1}{2}} + x^{\frac{1}{4}} = 116$, or $x^{\frac{1}{2}} + \frac{1}{7}x^{\frac{1}{4}} = \frac{116}{7}$:

$$\therefore\ x^{\frac{1}{2}} + \frac{1}{7}x^{\frac{1}{4}} + \left(\frac{1}{14}\right)^2 = \frac{116}{7} + \frac{1}{196}$$

$$= \frac{3249}{196}:$$

$$\text{whence, } x^{\frac{1}{4}} + \frac{1}{14} = \pm\frac{57}{14}, \text{ and } x^{\frac{1}{4}} = 4 \text{ or } -\frac{29}{7}:$$

$$\therefore\ x = 4^4 = 256, \text{ or } x = \left(-\frac{29}{7}\right)^4 = \frac{707281}{2401} = 294\frac{1387}{2401}.$$

If this equation had first been reduced to a rational form, the same values of x would have been obtained.

149. Many other equations, which by the ordinary reductions, would rise to higher dimensions than what are properly comprised in this class, may be solved by completing the square, provided they can be made to assume the form,

$$y^2 - Py = Q:$$

where y involves the unknown symbol, or its powers and roots, combined with known quantities: but as no general directions

can be given in addition to what have already been laid down, the student must rely upon his own judgment in the choice of the process which he may adopt.

Ex. 1. Given $2x^2 + \sqrt{2x^2+1} = 11$, to find x.

Adding 1 to both sides of the equation, we have

$$(2x^2+1) + \sqrt{2x^2+1} = 12:$$

$\therefore$ considering $2x^2+1$ in the place of the simple symbol y, we shall have

$$(2x^2+1) + \sqrt{2x^2+1} + \frac{1}{4} = \frac{49}{4}:$$

$$\therefore \sqrt{2x^2+1} + \frac{1}{2} = \pm\frac{7}{2}, \text{ and } \sqrt{2x^2+1} = 3 \text{ or } -4:$$

$$\text{whence, } 2x^2+1 = 9 \text{ or } 16, \quad x^2 = 4 \text{ or } \frac{15}{2}:$$

and the values of x are ± 2, and $\pm\frac{1}{2}\sqrt{30}$, two rational and two irrational roots.

Ex. 2. Given $\sqrt[m]{(1+x)^2} - \sqrt[m]{(1-x)^2} = \sqrt[m]{1-x^2}$, to find x.

Dividing both members by $\sqrt[m]{(1-x)^2}$, we have

$$\left(\frac{1+x}{1-x}\right)^{\frac{2}{m}} - 1 = \left(\frac{1+x}{1-x}\right)^{\frac{1}{m}}, \text{ or } \left(\frac{1+x}{1-x}\right)^{\frac{2}{m}} - \left(\frac{1+x}{1-x}\right)^{\frac{1}{m}} = 1:$$

$$\text{whence, } \left(\frac{1+x}{1-x}\right)^{\frac{2}{m}} - \left(\frac{1+x}{1-x}\right)^{\frac{1}{m}} + \frac{1}{4} = \frac{5}{4}:$$

$$\therefore \left(\frac{1+x}{1-x}\right)^{\frac{1}{m}} - \frac{1}{2} = \pm\frac{1}{2}\sqrt{5}, \text{ and } \left(\frac{1+x}{1-x}\right)^{\frac{1}{m}} = \frac{1}{2}(1\pm\sqrt{5}):$$

$$\text{whence, } \frac{1+x}{1-x} = \frac{(1\pm\sqrt{5})^m}{2^m}, \text{ two simple equations,}$$

$$\text{which give } x = \frac{(1\pm\sqrt{5})^m - 2^m}{(1\pm\sqrt{5})^m + 2^m}.$$

150. By getting rid of factors common to both the members of equations, the solutions of equations of higher dimensions may frequently be obtained by the preceding principles, but they will generally be subject to the imperfection mentioned in article (136).

Ex. Given $x^3 - 7x = 6$, to find x.

By adding $6x$ to both members, we have

$$x^3 - x = 6x + 6, \text{ or } x(x^2 - 1) = 6(x + 1):$$

whence, dividing both sides by $x + 1$, we obtain

$$x(x - 1) = 6, \text{ or } x^2 - x = 6:$$

from which the values of x are found to be 3 and -2: but in this example, there is another root $= -1$, which has been lost sight of in consequence of the equal division by $x + 1$, the proposed equation being equivalent to

$$(x + 1)(x + 2)(x - 3) = 0,$$

which is satisfied by making $x = -1$, $x = -2$, or $x = 3$.

151. If in order to avoid fractions, we leave a quadratic equation in the form,

$$ax^2 - bx = c:$$

and multiply every term by four times the coefficient of the first term, and add to both sides, the square of that of the second, we shall have

$$4a^2x^2 - 4abx + b^2 = 4ac + b^2,$$

whereof the former member is a complete square:

$$\therefore\ 2ax - b = \pm\sqrt{4ac + b^2}, \text{ and } x = \frac{b \pm \sqrt{4ac + b^2}}{2a}.$$

Ex. Given $7x^2 - 13x = 2$, to find x.

Multiplying all the terms by 4×7 or 28, we have

$$196x^2 - 364x = 56:$$

and completing the square by adding 13^2 or 169, we obtain

$$196x^2 - 364x + 169 = 225:$$

$$\text{whence,}\quad 14x - 13 = \pm 15,\quad \text{and}\quad x = 2 \text{ or } -\frac{1}{7}:$$

which are the same as would have been found by the ordinary process.

This is the *Hindoo* method of solving quadratic equations, and may be found in the *Beej Gunnit*, or *Beja Ganita* of *Bhasker Acharij*, a famous Mathematician who lived about the beginning of the *thirteenth* century.

PROBLEMS IN QUADRATIC EQUATIONS.

152. By proper attention to the observations and directions contained in article (137), problems belonging to this head will be algebraically exhibited by means of equations of the second degree, or such as are reducible to the second order by substitutions or other artifices: and it then remains only to apply the rules above laid down, to obtain their solutions.

As, however, every equation of the second order admits of two solutions at least, of which it frequently happens that only one will answer the conditions of the problem, it may be observed that in the process of reducing the equation to the proper form, a new condition is sometimes introduced, and a corresponding new value of the unknown symbol, which did not originally belong to it: or that the algebraical equation is more comprehensive than the enunciation of the problem, so that it comprises other conditions besides those which are peculiar to the question under consideration: and consequently it will be necessary at last to reject such values as are excluded by the nature of the case, as will be seen in some of the following examples.

Ex. 1. Find two numbers, one of which is three times as great as the other, and the sum of whose squares is 90.

$$\text{Let } x = \text{the less number}: \therefore\ 3x = \text{the greater}:$$

$$\text{and } x^2 + 9x^2 = 90, \text{ or } 10x^2 = 90, \text{ by the question}:$$

$$\text{whence, } x^2 = 9, \quad x = \pm 3, \text{ and } \therefore\ 3x = \pm 9:$$

so that in the view of common arithmetic, the numbers required are 3 and 9: and the negative numbers -3 and -9, which equally satisfy the equation, are merely the results of the *generalisation* of the arithmetical rules in Symbolical Algebra.

Ex. 2. Divide 12 into two parts, so that the square of one of them, may be four times as great as the square of the other.

$$\text{Let } x = \text{one of the parts}: \therefore\ 12 - x = \text{the other}:$$

$$\text{whence, } x^2 = 4(12-x)^2, \text{ by the question}:$$

$$\therefore\ x = \pm 2(12 - x) = 24 - 2x, \text{ or } = -24 + 2x:$$

$$\text{the former gives } x = 8, \text{ and } \therefore\ 12 - x = 4,$$

which are the arithmetical solution:

$$\text{the latter gives } x = 24, \text{ and } \therefore\ 12 - x = -12,$$

which answer the symbolical condition.

Ex. 3. Divide 20 into two parts, so that the product of the whole and one of the parts, may be equal to the square of the other part.

$$\text{Let } x = \text{one part}: \therefore\ 20 - x = \text{the other}:$$

$$\text{whence, } 20x = (20-x)^2 = 400 - 40x + x^2:$$

$$\therefore\ x^2 - 60x = -400, \text{ and } x^2 - 60x + 900 = 500:$$

$$\text{which gives } x - 30 = \pm 10\sqrt{5}, \text{ and } x = 30 \pm 10\sqrt{5}:$$

$$\text{that is, if } x = 30 + 10\sqrt{5}, \text{ then } 20 - x = -10 - 10\sqrt{5}:$$

$$\text{and if } x = 30 - 10\sqrt{5}, \text{ then } 20 - x = -10 + 10\sqrt{5}.$$

The former of these solutions is evidently symbolical: and the latter, which makes the required parts to be $30 - 10\sqrt{5}$

and $10\sqrt{5}-10$, shews that the proposed parts cannot be whole numbers, or even rational quantities, which alone are the *primary* objects of arithmetical computation.

Ex. 4. Divide a into two parts, whose product shall be b^2.

Let x = one part, and therefore $a-x$ = the other: whence,

$$x(a-x), \text{ or } ax-x^2=b^2, \text{ by the question:}$$

and this solved, gives $x=\frac{1}{2}(a\pm\sqrt{a^2-4b^2})$: that is,

$$x=\frac{1}{2}(a\pm\sqrt{a^2-4b^2}) \text{ and } a-x=\frac{1}{2}(a\mp\sqrt{a^2-4b^2}),$$

are the parts required: and it is observable that the two parts are the same, whether the upper or lower sign of the radical quantity be used.

The forms of these results enable us to assign the limits under which the problem is *possible:* for it is evident that if $4b^2$ be greater than a^2, $\sqrt{a^2-4b^2}$ becomes *imaginary*, and thus the two parts are unassignable, according to the principles of Arithmetic; that is, no such parts can be found: and the extreme possible case will manifestly be, when $\sqrt{a^2-4b^2}=0$, so that $x=\frac{1}{2}a$, $a-x=\frac{1}{2}a$, and $b^2=\frac{1}{4}a^2$ is the greatest possible, as in article (46).

Ex. 5. If the sum of two magnitudes be $2a$, and the sum of their cubes $2b^3$: find them.

For the purpose of avoiding equations of a higher order than the second, let $2x$ represent the difference of the required magnitudes: then, by article (44), they are $a+x$ and $a-x$:

$$\therefore\ (a+x)^3+(a-x)^3=2b^3, \text{ or } a^3+3ax^2=b^3,$$

which gives

$$a+x=a+\sqrt{\frac{b^3-a^3}{3a}}, \text{ and } a-x=a-\sqrt{\frac{b^3-a^3}{3a}}.$$

In order that these magnitudes may be assignable, it is manifest that b must not be less than a: and the extreme case which the problem is capable of an arithmetical solution, be when $b=a$, and therefore the magnitudes are equal.

Ex. 6. Of a number of bees, after eight-ninths and the square root of half of them, had flown away, there were two remaining: what was the number at first?

For the sake of avoiding irrational forms, let $2x^2$ represent the number of bees at first: then,

$$\frac{16x^2}{9} + x + 2 = 2x^2, \text{ by the question:}$$

$$\therefore\ 16x^2 - 72x + 81 = 225, \text{ and } 4x - 9 = \pm 15:$$

from which the values of x are found to be 6, and $-1\frac{1}{2}$: and the latter value being excluded by the nature of the case, the number of bees must be $2 \times 6^2 = 72$.

Ex. 7. A farmer purchased a number of oxen for £112, and observed that if he had had one more for the same money, each of them would have cost him £2. less: required the number he purchased, and the price of each.

Let x = the number of oxen purchased;

$$\therefore\ \frac{112}{x} = \text{the price of each:}$$

also, according to the farmer's observation, if the number purchased had been $x + 1$, the price of each would have been $\frac{112}{x} - 2$: whence, $(x + 1)\left(\frac{112}{x} - 2\right) = 112$, by the question: and this, by reduction becomes

$$x^2 + x = 56:$$

which solved, gives $x = 7$, and $x = -8$.

Hence, from the problem considered only in an arithmetical point of view, we find the number of oxen to be 7, and therefore the price of each will be $\frac{112}{7} = £16$.

In this instance -8 the second value of x, as found above, though it does not fulfil the *condition* of the problem, at the same time that it satisfies the *equation* expressing it, is never-

theless not without a significant meaning, as we shall now shew: for, the negative symbol $-x$ being supposed to represent the negative root -8, the algebraical expression of the problem becomes

$$(-x+1)\left(-\frac{112}{x}-2\right)=112, \text{ or } (x-1)\left(\frac{112}{x}+2\right)=112,$$

and this being again translated out of symbolical, into common language, manifestly gives the following problem:

"A farmer purchased a number of oxen for £112, and observed that if he had had one *fewer* for the same money, each of them would have cost him £2. *more:* required the number he purchased, and the price of each;"

so that each of the algebraical equations above found, comprehending both these problems, is more general than the enunciation of either, and at the same time, furnishes an arithmetical and symbolical solution of the two.

SIMULTANEOUS EQUATIONS.

153. DEF. From what has been already said, it is evident that the value of any one of the symbols concerned in an equation is entirely dependent upon the values of the rest, and it can become *known*, only when those of the rest are *given* or *assigned*.

If then there exist at the same time, two or more equations established among the same symbols, it is obvious that the determination of any one of them cannot be effected without reference to the assignability of the rest: but the processes already explained will generally enable us to make such modifications and changes, that one or more of the unknown quantities shall not be found in the equations which result.

Quantities thus treated are said to be *eliminated* or *exterminated:* and when the numbers of equations and unknown quantities are properly adjusted, it is manifest that the given equations may all be reduced to one final equation comprising only one unknown symbol.

Equations thus connected with each other are termed *Simultaneous Equations*, and it will shortly appear that two equations not derivable from each other by the fundamental operations of Arithmetic, are *necessary* and *sufficient* for the determination of two unknown quantities: three equations for that of three unknown quantities: and so on.

We will illustrate this by the following methods usually resorted to for the solution of equations of this description.

154. *First Method.* When there are two equations involving two unknown symbols, which are supposed to be *identical* in both, it is evident that if either symbol be expressed in terms of the other by means of both equations, and its values be then equated to each other, the equation resulting will comprise only one unknown quantity, and may be solved by the principles above explained.

Ex. 1. Given $x+y=9$, and $3x+5y=35$, to find x and y.

Representing these equations in order by (1) and (2), we have from (1), $x=9-y$: from (2), $x=\dfrac{35-5y}{3}$: whence,

$9-y=\dfrac{35-5y}{3}$, an equation with one unknown quantity:

$$\therefore\ 27-3y=35-5y,\quad 2y=8,\ \text{ and }\ y=4:$$

$$\therefore\ \text{from (1), we find }\ x=9-y=9-4=5:$$

that is, $x=5$, and $y=4$ fulfil simultaneously both the equations proposed.

Ex. 2. Find the values of x and y in the equations:

$$\frac{x+2}{7}+\frac{y-x}{4}=2x-8,\ \text{ and }\ \frac{2y-3x}{3}+2y=3x+4.$$

Clearing of fractions and reducing as much as possible, we have

$$\text{from (1), } 59x-7y=232, \text{ and } \therefore\ x=\frac{232+7y}{59}:$$

from (2), $3x - 2y = -3$, and $\therefore\ x = \dfrac{2y-3}{3}$:

whence, $\dfrac{232+7y}{59} = \dfrac{2y-3}{3}$, or $696 + 21y = 118y - 177$:

$$\therefore\ 97y = 873, \text{ and } y = \frac{873}{97} = 9:$$

$$\therefore \text{ from (2), } x = \frac{2y-3}{3} = \frac{18-3}{3} = \frac{15}{3} = 5:$$

that is, $x = 5$ and $y = 9$ are the values required.

Ex. 3. Given $x + y = 34$, and $x - \sqrt{xy} = 10$, to find x and y.

From (1), $y = 34 - x$: from (2), $y = \left(\dfrac{x-10}{\sqrt{x}}\right)^2$:

$$\therefore\ 34 - x = \left(\frac{x-10}{\sqrt{x}}\right)^2 = \frac{x^2 - 20x + 100}{x}:$$

whence, $34x - x^2 = x^2 - 20x + 100$, or $2x^2 - 54x = -100$:

$\therefore\ x^2 - 27x = -50$, an equation of one unknown quantity:

and completing the square, &c., we have

$$x^2 - 27x + \left(\frac{27}{2}\right)^2 = \left(\frac{27}{2}\right)^2 - 50 = \frac{529}{4}:$$

$$\therefore\ x - \frac{27}{2} = \pm\frac{23}{2}, \text{ and } x = \pm\frac{23}{2} + \frac{27}{2} = \frac{50}{2} = 25, \text{ or } = \frac{4}{2} = 2:$$

also, from (1), $y = 34 - x = 34 - 25 = 9$, or $y = 34 - 2 = 32$:

$$\therefore\ x = 25, \text{ when } y = 9, \text{ and } x = 2, \text{ when } y = 32:$$

that is, $\left.\begin{array}{r} x = 25 \\ y = 9 \end{array}\right\}$, and $\left.\begin{array}{r} x = 2 \\ y = 32 \end{array}\right\}$, are

simultaneous solutions of the equations proposed, the latter being *symbolical*, because $\sqrt{xy}$ must be supposed to be negative, before these numbers will satisfy them.

In examples of this description, it is absolutely necessary to keep the values of the symbols distinct from each other: for it is readily seen that $x=25$ and $y=32$, do not satisfy the proposed equations, and therefore do not express *simultaneous* or *corresponding* values of the unknown symbols.

155. *Second Method.* When there are two equations involving two unknown symbols, if the value of one symbol be expressed in terms of the other by means of either equation, this value substituted in the other equation, will manifestly reduce it so as to involve only one unknown quantity, to which the usual methods may be applied.

Ex. 1. Given $7x+10y=41$, and $13x-11y=17$, to find x and y.

From (1), $x=\dfrac{41-10y}{7}$: and by substituting this for x in (2), we have

$$13\left(\frac{41-10y}{7}\right)-11y=17:$$

$$\therefore\ 533-207y=119, \text{ by reduction}:$$

$$\text{whence, } 207y=414, \text{ and } \therefore\ y=\frac{414}{207}=2:$$

and from (1), we have

$$x=\frac{41-10y}{7}=\frac{41-20}{7}=\frac{21}{7}=3:$$

that is, $x=3$ and $y=2$, satisfy both the equations.

Ex. 2. Given $x^2+xy=66$, and $x^2-y^2=11$, to find x and y.

From (1), $y=\dfrac{66-x^2}{x}$, and $\therefore\ y^2=\left(\dfrac{66-x^2}{x}\right)^2$:

$\therefore$ by substitution in (2), we obtain

$$x^2-\left(\frac{66-x^2}{x}\right)^2=11, \text{ or } 4356-132x^2+x^4=x^4-11x^2:$$

whence, $121x^2 = 4356$, $x^2 = 36$, and $x = \pm 6$:

also, from (2), $y^2 = x^2 - 11 = 36 - 11 = 25$, and $\therefore\ y = \pm 5$:

and here, $\left.\begin{matrix} x = 6 \\ y = 5 \end{matrix}\right\}$, $\left.\begin{matrix} x = -6 \\ x = -5 \end{matrix}\right\}$, are the simultaneous solutions, as may easily be verified.

Ex. 3. Given $2x = 3y$, and $x^3 - y^3 = 19$, to find x and y.

From (1), $y = \frac{2}{3}x$: and by substitution in (2), we have

$$x^3 - \frac{8}{27}x^3 = 19:$$

that is, $27x^3 - 8x^3 = 19 \times 27$, or $19x^3 = 19 \times 27$:

$\therefore\ x^3 = 27$, and $x = 3$: also, $y = \frac{2}{3}x = \frac{2}{3}(3) = 2$.

whence, $x = 3$, and $y = 2$ satisfy both equations.

156. *Third Method.* When two equations involving two unknown quantities, are reduced to their simplest forms, and the coefficients of either symbol are rendered, if necessary, identical in both, this symbol may evidently be exterminated by addition or subtraction of the corresponding members of the equations, according as it has different or the same algebraical signs in both.

Ex. 1. Given $2x + 9y = 20$, and $11x - 3y = 5$, to find x and y.

From (1), $2x + 9y = 20$:

from (2), $33x - 9y = 15$, by multiplying by 3:

$\therefore$ by addition, $35x = 35$, and $x = 1$:

from (1), $22x + 99y = 220$, by multiplying by 11:

from (2), $22x - 6y = 10$, by multiplying by 2.

$\therefore$ by subtraction, $105y = 210$, and $y = 2$:

whence, $x = 1$, and $y = 2$ are the required answers.

Ex. 2. Find the values of x and y from the equations:

$$\frac{4}{x}+\frac{5}{y}=\frac{9}{y}-1, \text{ and } \frac{5}{x}+\frac{4}{y}=\frac{7}{x}+\frac{3}{2}.$$

From (1), $\frac{4}{x}=\frac{4}{y}-1$, and from (2), $\frac{2}{x}=\frac{4}{y}-\frac{3}{2}$:

$$\therefore \frac{4}{x}=\frac{4}{y}-1, \text{ from (1)}:$$

$$\text{and } \frac{4}{x}=\frac{8}{y}-3, \text{ from (2)}:$$

whence, by subtracting the upper from the lower, we have

$$0=\frac{4}{y}-2, \quad \frac{4}{y}=2, \quad 4=2y, \text{ and } y=2:$$

$$\text{also, } \frac{4}{x}=\frac{4}{y}-1=\frac{4}{2}-1=2-1=1, \text{ and } x=4.$$

In such instances as this, when we have found the values of the reciprocals of x and y, those of x and y immediately become known.

Ex. 3. Given $ax+by=c^2$, and $\frac{a}{b+y}-\frac{b}{a+x}=0$, to find x and y.

From (2), $a^2+ax-b^2-by=0$:

from (1), $\qquad ax \qquad +by=c^2$:

and adding together the corresponding members, we obtain

$$a^2+2ax-b^2=c^2, \text{ or } x=\frac{b^2+c^2-a^2}{2a}:$$

also, subtracting the upper from the lower, we have

$$-a^2+b^2+2by=c^2, \text{ or } y=\frac{a^2-b^2+c^2}{2b}:$$

and it may be worth while to verify these results.

157. In a great variety of equations that are met with involving two unknown quantities, the ordinary processes hitherto described may frequently be dispensed with, and recourse be had to expedients which will in most cases abridge the labour of solution. The discovery of such expedients must however be left to the student's ingenuity, and some of the most general are resorted to in the following examples.

Ex. 1. Given $x^2 + y^2 = 52$, and $xy = 24$, to find x and y.

Here, by addition, subtraction, &c. we have

$$x^2 + 2xy + y^2 = 100, \quad x^2 - 2xy + y^2 = 4:$$

$$\therefore \; x + y = \pm 10, \text{ and } x - y = \pm 2:$$

$$\text{whence, we find } 2x = \pm 12, \text{ and } x = \pm 6:$$

$$\text{also, } 2y = \pm 8, \text{ and } y = \pm 4:$$

that is, $\left.\begin{matrix} x = 6 \\ y = 4 \end{matrix}\right\}, \quad \left.\begin{matrix} x = -6 \\ y = -4 \end{matrix}\right\}$, are the corresponding solutions.

Ex. 2. Given $x^2 - xy = a^2$, and $xy - y^2 = b^2$, to find x and y.

From (1), $x(x - y) = a^2$: from (2), $y(x - y) = b^2$:
whence, dividing the latter by the former, we obtain

$$\frac{y}{x} = \frac{b^2}{a^2}, \text{ and } y = \frac{b^2}{a^2}x:$$

$\therefore$ by substituting this value of y in (1), we find

$$x^2 - \frac{b^2}{a^2}x^2 = a^2, \text{ or } (a^2 - b^2)x^2 = a^4:$$

$$\text{whence, } x = \pm \frac{a^2}{\sqrt{a^2 - b^2}}, \text{ and } y = \frac{b^2}{a^2}x = \pm \frac{b^2}{\sqrt{a^2 - b^2}}.$$

Ex. 3. Given $x^3y^2 + xy^4 = 156$, and $2x^3y^2 - x^2y^3 = 144$, to find x and y.

Here, $xy^4 + x^2y^3 - x^3y^2 = 12$, by subtraction:

that is, $xy^2(y^2 + xy - x^2) = 12 = \frac{1}{13}xy^2(x^2 + y^2)$:

$$\therefore\ 13y^2 + 13xy - 13x^2 = x^2 + y^2:$$

and $12y^2 + 13xy = 14x^2$, or $y^2 + \frac{13x}{12}y = \frac{7}{6}x^2$:

whence, completing the square in y, we obtain

$$y^2 + \frac{13x}{12}y + \frac{169x^2}{576} = \frac{7x^2}{6} + \frac{169x^2}{576} = \frac{841x^2}{576}:$$

$$\therefore\ y + \frac{13}{24}x = \pm\frac{29}{24}x, \text{ and } y = \frac{2}{3}x, \text{ or } -\frac{7}{4}x:$$

assuming $y = \frac{2}{3}x$, and substituting in (1), we have

$$x^5\left(\frac{4}{9} + \frac{16}{81}\right) = 156, \text{ or } \frac{52}{81}x^5 = 156, \text{ and } \therefore\ x = 3:$$

whence is obtained $y = \frac{2}{3}x = 2$:

again, if we take $y = -\frac{7}{4}x$, the same operation gives

$$x^5\left(\frac{49}{16} + \frac{2401}{256}\right) = 156, \text{ or } \frac{3185}{256}x^5 = 156:$$

whence, $x = \sqrt[5]{\frac{3072}{245}}$, and $\therefore\ y = -\frac{7}{4}x = -\frac{7}{4}\sqrt[5]{\frac{3072}{245}}$.

Ex. 4. Given $3x^2 - 4xy + y^2 = 20$, and $2x^2 - 7y^2 = 4$, to find x and y.

Here, $3x^2 - 4xy + y^2 = 20 = 4 \times 5 = 10x^2 - 35y^2$:

$$\therefore\ 36y^2 - 4xy = 7x^2, \text{ and } y^2 - \frac{1}{9}xy = \frac{7}{36}x^2:$$

completing the square, &c. we obtain $y = \frac{1}{2}x$, or $-\frac{7}{18}x$:

substituting these values in (2), we have

$$2x^2 - \frac{7}{4}x^2 = 4, \ x^2 = 16, \ x = \pm 4, \text{ and } \therefore \ y = \pm 2:$$

$$\text{also, } 2x^2 - \frac{343}{324}x^2 = 4, \ \therefore \ x = \pm\frac{36}{\sqrt{305}}, \text{ and } y = \mp\frac{14}{\sqrt{305}}.$$

Ex. 5. Given $x + y + \frac{y^2}{x} = 20$, and $x^2 + y^2 + \frac{y^4}{x^2} = 140$, to find x and y.

Dividing the members of the latter equation by the corresponding members of the former, we have

$$x - y + \frac{y^2}{x} = 7:$$

$$\text{also, } x + y + \frac{y^2}{x} = 20:$$

$$\therefore \text{ by subtraction, } 2y = 13, \text{ and } y = 6\tfrac{1}{2}:$$

$$\text{also, by addition, } 2x + \frac{2y^2}{x} = 27, \text{ or } 2x + \frac{169}{2x} = 27:$$

$$\therefore \ 4x^2 - 54x = -169, \text{ and } x = \tfrac{1}{4}(27 \pm \sqrt{53}):$$

$$\text{so that, } \left.\begin{matrix} x = \tfrac{1}{4}(27 + \sqrt{53}) \\ y = 6\tfrac{1}{2} \end{matrix}\right\}, \quad \left.\begin{matrix} x = \tfrac{1}{4}(27 - \sqrt{53}) \\ y = 6\tfrac{1}{2} \end{matrix}\right\},$$

are the solutions of the equations.

Ex. 6. Find the values of x and y from the equations: $(x+y)^{-2} - (x-y)^{-2} = a^{-2} - b^{-2}$, and $a^{-2}(x+y)^2 + b^{-2}(x-y)^2 = 1$.

Multiplying together the corresponding members, we have

$$\frac{1}{a^2} - \frac{1}{b^2} - \frac{1}{a^2}\left(\frac{x+y}{x-y}\right)^2 + \frac{1}{b^2}\left(\frac{x-y}{x+y}\right)^2 = \frac{1}{a^2} - \frac{1}{b^2}:$$

$$\therefore \ \frac{1}{a^2}\left(\frac{x+y}{x-y}\right)^2 = \frac{1}{b^2}\left(\frac{x-y}{x+y}\right)^2, \text{ or } \frac{1}{a}\left(\frac{x+y}{x-y}\right) = \frac{1}{b}\left(\frac{x-y}{x+y}\right):$$

$$\therefore \ (x-y)^2 = \frac{b}{a}(x+y)^2: \text{ and by substitution in (2),}$$

we find $\dfrac{(x+y)^2}{a^2} + \dfrac{(x+y)^2}{ab} = 1$, or $(x+y)^2 = \dfrac{a^2 b}{a+b}$:

whence, $x + y = \pm \dfrac{a\sqrt{b}}{\sqrt{a+b}}$, and $x - y = \pm \dfrac{b\sqrt{a}}{\sqrt{a+b}}$:

$$\therefore\ x = \pm \frac{a\sqrt{b} + b\sqrt{a}}{2\sqrt{a+b}} = \pm \frac{\sqrt{ab}\,(\sqrt{a} + \sqrt{b})}{2\sqrt{a+b}}:$$

$$y = \pm \frac{a\sqrt{b} - b\sqrt{a}}{2\sqrt{a+b}} = \pm \frac{\sqrt{ab}\,(\sqrt{a} - \sqrt{b})}{2\sqrt{a+b}}:$$

which are both real, whatever be the relative values of a and b.

If $\dfrac{1}{a}\left(\dfrac{x+y}{x-y}\right) = -\dfrac{1}{b}\left(\dfrac{x-y}{x+y}\right)$, we have $(x-y)^2 = -\dfrac{b}{a}(x+y)^2$:

and by substitution in (2) as before, we obtain

$$\frac{(x+y)^2}{a^2} - \frac{(x+y)^2}{ab} = 1, \text{ or } (x+y)^2 = \frac{a^2 b}{b-a}:$$

$$\therefore\ x + y = \pm \frac{a\sqrt{-b}}{\sqrt{a-b}}, \text{ and } x - y = \pm \frac{b\sqrt{a}}{\sqrt{a-b}}:$$

whence, $x = \pm \dfrac{a\sqrt{-b} + b\sqrt{a}}{2\sqrt{a-b}} = \pm \dfrac{\sqrt{ab}\,(\sqrt{-a} + \sqrt{b})}{2\sqrt{a-b}}$:

$$y = \pm \frac{a\sqrt{-b} - b\sqrt{a}}{2\sqrt{a-b}} = \pm \frac{\sqrt{ab}\,(\sqrt{-a} - \sqrt{b})}{2\sqrt{a-b}},$$

which are both imaginary, whether a be greater or less than b.

Ex. 7. Find the values of x and y, in the equations:

$x + \sqrt{3y^2 - 11 + 2x} = 7 + 2y - y^2$, and $\sqrt{3y - x + 7} = \dfrac{x+y}{x-y}$.

From (1), $y^2 + x - 7 + \sqrt{3y^2 + 2x - 11} = 2y$:

$$\therefore\ 2y^2 + 2x - 14 + 2\sqrt{3y^2 + 2x - 11} = 4y:$$

$$\therefore\ (3y^2 + 2x - 11) + 2\sqrt{3y^2 + 2x - 11} = y^2 + 4y + 3:$$

whence, completing the square according to article (149), and extracting the root, we have

$$\sqrt{3y^2 + 2x - 11} + 1 = \pm (y + 2):$$

using the upper sign, we obtain

$$3y^2 + 2x - 11 = y^2 + 2y + 1: \text{ and } x = 6 + y - y^2:$$

whence, $3y - x + 7 = 3y - 6 - y + y^2 + 7 = y^2 + 2y + 1:$

$$\therefore\ \frac{x + y}{x - y} = \sqrt{3y - x + 7} = y + 1:$$

$$\therefore\ xy = y^2 + 2y, \text{ and } x = y + 2:$$

and equating the two expressions for x, we have

$$y + 2 = 6 + y - y^2,\ y^2 = 4,\ y = 2, \text{ and } \therefore\ x = 4.$$

There are evidently many other simultaneous values of x and y which will satisfy the equations, but it would be no easy matter to find them all.

158. When there are three independent equations subsisting among three algebraical symbols, the preceding principles shew that by means of two of them, any two of the unknown quantities may be expressed in terms of the remaining one; and if these values be substituted in the third equation, it will manifestly be reduced to an equation with only one unknown symbol, which may be treated as before: and thus it follows that in general, three independent equations will be both *necessary* and *sufficient* for the determination of three unknown quantities.

Ex. 1. Given $x + 2y + 3z = 14$, $2x - 3y + 4z = 8$, and $3x + 4y - 5z = -4$, to find x, y and z.

From (1), $x = 14 - 2y - 3z$: from (2), $x = \dfrac{8 + 3y - 4z}{2}$:

$$\therefore\ 14 - 2y - 3z = \frac{8 + 3y - 4z}{2}, \text{ and } 28 - 4y - 6z = 8 + 3y - 4z:$$

$$\text{whence, } y = \frac{20 - 2z}{7}, \text{ and } \therefore\ x = \frac{58 - 17z}{7}:$$

$\therefore$ by substitution in (3), we obtain

$$\frac{3}{7}(58 - 17z) + \frac{4}{7}(20 - 2z) - 5z = -4,$$

which being reduced and solved, gives $z = 3$:

$$\text{whence, } x = \frac{58 - 17z}{7} = 1, \text{ and } y = \frac{20 - 2z}{7} = 2:$$

that is, $x = 1$, $y = 2$ and $z = 3$, satisfy the equation.

This method of solution has been here adopted in order to evince the truth of the principle asserted in the article, but other methods will be more convenient in particular cases.

Ex. 2. Find x, y and z by means of the following equations:

$$x - ay + a^2z - a^3 = 0;$$
$$x - by + b^2z - b^3 = 0:$$
$$x - cy + c^2z - c^3 = 0.$$

From (1), $x = ay - a^2z + a^3$:

(2), $x = by - b^2z + b^3$:

(3), $x = cy - c^2z + c^3$:

$\therefore\ ay - a^2z + a^3 = by - b^2z + b^3$: from (1) and (2):

$ay - a^2z + a^3 = cy - c^2z + c^3$: from (1) and (3):

$\therefore\ (a - b)y = (a^2 - b^2)z - (a^3 - b^3)$:

and $y = (a + b)z - (a^2 + ab + b^2)$: (α)

$\therefore\ (a - c)y = (a^2 - c^2)z - (a^3 - c^3)$:

and $y = (a + c)z - (a^2 + ac + c^2)$: (β)

whence, equating the values of y in (α) and (β), we have

$$(a + b)z - (a^2 + ab + b^2) = (a + c)z - (a^2 + ac + c^2):$$

$\therefore\ (b - c)z = a(b - c) + (b^2 - c^2)$, and $z = a + b + c$:

also, from (α), we obtain

$$y = (a + b)(a + b + c) - (a^2 + ab + b^2) = ab + ac + bc:$$

and from (1), we find

$$x = a(ab + ac + bc) - a^2(a + b + c) + a^3 = abc.$$

Ex. 3. Given $bz + cy = a$, $az + cx = b$, and $ay + bx = c$, to find x, y and z.

$$\text{From (1),} \quad abz + acy = a^2:$$
$$\text{(2),} \quad abz + bcx = b^2:$$
$$\text{(3),} \quad acy + bcx = c^2:$$

$\therefore$ by adding together the corresponding members, we have

$$2abz + 2acy + 2bcx = a^2 + b^2 + c^2:$$

$$\text{from (1),} \quad 2abz + 2acy \qquad = 2a^2:$$

$\therefore$ by subtraction of the corresponding members, we find

$$2bcx = b^2 + c^2 - a^2, \text{ and } \therefore x = \frac{b^2 + c^2 - a^2}{2bc}:$$

$$\text{similarly, } y = \frac{a^2 + c^2 - b^2}{2ac}, \text{ and } z = \frac{a^2 + b^2 - c^2}{2ab}.$$

159. It has been seen above that two independent equations are necessary and sufficient for the determination of two unknown quantities, and three equations for three unknown quantities: and the kind of reasoning employed will prove that n independent equations involving n unknown quantities are necessary and sufficient for assigning all their values.

If there be more equations than unknown quantities, some of them will evidently be *redundant*, and may be *inconsistent*: but if there be fewer, the values of all the symbols will be *indeterminate*, or be *assignable* only by *assuming* the values of the requisite number of them to be already *known*.

160. Before quitting this part of the subject, we will explain a principle much used in the higher applications of Algebra, and shew that *Elimination* in all cases amounts to *Partial Solution*.

From the nature of the arithmetical operations employed, it is evident that no symbol can be got rid of from an equation,

without replacing it by its value, ascertained by means of others with which it is supposed to be connected.

Ex. 1. If we have $y = a_1 x + b_1$, and $y = a_2 x + b_2$: by subtraction, we obtain

$$(a_2 - a_1) x + (b_2 - b_1) = 0:$$

and in this, y has virtually been replaced by its value in terms of x, ascertained by means of the two equations combined: and moreover, the value of y becomes known, when that of x is found, which may be done by means of the equation above: for,

$$x = -\frac{b_2 - b_1}{a_2 - a_1}.$$

Also, $$a_2 y = a_1 a_2 x + a_2 b_1:$$

$$a_1 y = a_1 a_2 x + a_1 b_2:$$

$$\therefore\ (a_2 - a_1) y = a_2 b_1 - a_1 b_2:$$

to which similar observations may be applied: and x is thus expressed in terms of y, the value of y being $= \dfrac{a_2 b_1 - a_1 b_2}{a_2 - a_1}$.

Ex. 2. If we have given the three following equations:

$$a_1 x + b_1 y + c_1 z = d_1:$$

$$a_2 x + b_2 y + c_2 z = d_2:$$

$$a_3 x + b_3 y + c_3 z = d_3:$$

then, $a_1 a_2 x + a_2 b_1 y + a_2 c_1 z = a_2 d_1$, from (1):

and $a_1 a_2 x + a_1 b_2 y + a_1 c_2 z = a_1 d_2$, from (2):

whence, by subtraction, we obtain

$$(a_2 b_1 - a_1 b_2) y + (a_2 c_1 - a_1 c_2) z = a_2 d_1 - a_1 d_2:$$

an equation involving only two unknown quantities: and the one which has disappeared, must evidently have been replaced by its proper value in terms of them.

Similarly, $(a_3 b_1 - a_1 b_3) y + (a_3 c_1 - a_1 c_3) z = a_3 d_1 - a_1 d_3$.

Representing these two equations, for brevity's sake, by

$$B_1 y + C_1 z = D_1, \text{ and } B_2 y + C_2 z = D_2,$$

and proceeding as above: we shall have

$$B_1 B_2 y + B_2 C_1 z = B_2 D_1 :$$

$$B_1 B_2 y + B_1 C_2 z = B_1 D_2 :$$

whence, by subtraction, we find

$$(B_2 C_1 - B_1 C_2) z = B_2 D_1 - B_1 D_2,$$

an equation involving only one unknown quantity, so that the two others have in reality been expressed in terms of the one which remains.

Finally, we have $z = \dfrac{B_2 D_1 - B_1 D_2}{B_2 C_1 - B_1 C_2}$

$$= \frac{(a_3 b_1 - a_1 b_3)(a_2 d_1 - a_1 d_2) - (a_2 b_1 - a_1 b_2)(a_3 d_1 - a_1 d_3)}{(a_3 b_1 - a_1 b_3)(a_2 c_1 - a_1 c_2) - (a_2 b_1 - a_1 b_2)(a_3 c_1 - a_1 c_3)}:$$

and thence the values of x and y may be found in similar forms.

PROBLEMS IN SIMULTANEOUS EQUATIONS.

161. Any two or more of the letters x, y, z, &c. being assumed to represent the magnitudes that are required, and the question being translated into algebraical language as before directed, the number of independent equations thence arising will be the same as the number of unknown quantities, if the problem be determinate in its nature: and consequently, by the methods of resolving equations just given, we shall arrive at known values for these quantities, and thereby at the solution required.

Ex. 1. In a certain employment, nine men and seven women receive together £3. 11*s*. 2*d*. for their wages; and it is found that seven men receive 19*s*. 8*d*. more than five women: required the wages of each.

Let x and y represent the wages of a man and a woman in pence:

$$\begin{array}{l} \qquad\qquad\qquad \text{£. s. d.} \quad \text{d.} \\ \text{then, } \left.\begin{array}{r} 9x + 7y = 3.\ 11.\ 2. = 854 \\ 7x - 5y = 0.\ 19.\ 8. = 236 \end{array}\right\}, \text{ by the question:} \end{array}$$

$$\text{from (1), } 63x + 49y = 5978:$$

$$\text{from (2), } 63x - 45y = 2124:$$

$$\therefore \text{ by subtraction, } 94y = 3854, \text{ and } y = 41d. = 3s.\ 5d:$$

$$\therefore \text{ from (2), } 7x = 236 + 5y = 441, \text{ and } x = 63d. = 5s.\ 3d.:$$

that is, each man receives $5s.\ 3d.$, and each woman $3s.\ 5d.$

Ex. 2. *A* wishes to pay *B* the sum of £12., but has nothing but moidores, and *B* has nothing but guineas: and the total number of pieces which pass between them is 16: find the number of each.

Let x = the number of moidores *A* pays to *B*:

y = the number of guineas *B* repays to *A*:

$$\text{then, } x + y = 16, \text{ and } 27x - 21y = 240:$$

$$\therefore \text{ from (2), } 9x - 7y = 80:$$

$$\text{from (1), } 9x + 9y = 144:$$

$$\therefore 16y = 64, \text{ and } y = 4: \text{ and therefore } x = 12:$$

that is, *A* pays to *B* twelve moidores, and receives back four guineas, which evidently discharges the debt.

Ex. 3. A person distributes a shillings among n persons, giving p pence each to some, and q pence each to others: required the number of each.

Let x and y denote the numbers receiving p and q pence respectively: then, by the question,

$$x + y = n, \text{ and } px + qy = 12a:$$

$$\begin{array}{rl} \text{from (1),} & px + py = np \\ \text{(2),} & px + qy = 12a \\ \hline \end{array}$$

$$\therefore (p - q)\,y = np - 12a, \text{ and } y = \frac{np - 12a}{p - q}:$$

$$\text{from (1), } \quad qx + qy = nq$$
$$\text{(2), } \quad px + qy = 12a$$

$$\therefore \ (p - q)\,x = 12a - nq, \text{ and } x = \frac{12a - nq}{p - q}:$$

that is, the two numbers required are expressed by

$$\frac{12a - nq}{p - q} \text{ and } \frac{np - 12a}{p - q}.$$

In the first place, it is evident, that in order to be consistent with the natural conditions of the problem, both these quantities must be positive whole numbers: that is, if p be greater than q, both $12a - nq$ and $np - 12a$ must be exactly divisible by $p - q$, at the same time that they are both positive quantities: and to ensure this last condition, we must have

$$12a - nq > 0, \text{ and } np - 12a > 0:$$

$$\therefore \ nq < 12a, \text{ or } n < \frac{12a}{q}, \text{ and } np > 12a, \text{ or } n > \frac{12a}{p}:$$

that is, n must be intermediate in magnitude to $\frac{12a}{q}$ and $\frac{12a}{p}$: and unless this is the case, the solution is merely *symbolical*, and of no manner of use.

If p be less than q, or $p - q$ be negative, we must have

$$12a - nq < 0, \text{ and } np - 12a < 0:$$

$$\therefore \ nq > 12a, \text{ or } n > \frac{12a}{q}, \text{ and } np < 12a, \text{ or } n < \frac{12a}{p}:$$

that is, n must still lie between $\frac{12a}{q}$ and $\frac{12a}{p}$.

Ex. 4. Required two numbers whose product is equal to the difference of their squares, and the sum of whose squares is equal to the difference of their cubes.

Let x and xy represent the two numbers:

$$\text{then, } \quad x^2y = x^2y^2 - x^2, \text{ or } y = y^2 - 1:$$
$$x^2y^2 + x^2 = x^3y^3 - x^3, \text{ or } y^2 + 1 = xy^3 - x:$$

from (1), we have $y = \frac{1}{2}(1 \pm \sqrt{5})$:

from (2), we have $x = \dfrac{y^2 + 1}{y^3 - 1} = \dfrac{y + 2}{2y} = \dfrac{5 \pm \sqrt{5}}{2(1 \pm \sqrt{5})} = \pm \frac{1}{2}\sqrt{5}$:

that is, $x = \pm \frac{1}{2}\sqrt{5}$, and $xy = \pm \frac{1}{4}(\sqrt{5} \pm 5)$, are the magnitudes required: and these results shew that there are no *numbers* properly so called, which satisfy the conditions of the question, though *real magnitudes* have been found which answer the purpose.

Ex. 5. Find two magnitudes, whose sum, product, and the sum of whose squares are equal to each other.

Let x and y represent the required magnitudes:

$$\text{then, we have } x + y = xy = x^2 + y^2:$$

from (1), $x = \dfrac{y}{y - 1}$, and from (2), $x = \frac{1}{2}y(1 \pm \sqrt{-3})$:

$$\therefore \frac{1}{y - 1} = \frac{1 \pm \sqrt{-3}}{2}, \text{ which gives } y = \tfrac{1}{2}(3 \mp \sqrt{-3}):$$

$$\therefore x = \frac{y}{y - 1} = \tfrac{1}{2}(3 \pm \sqrt{-3}):$$

and these imaginary values of x and y, which will be found to satisfy both the equations expressing the specified properties, prove that there exist no magnitudes whatever, which, consistently with *any* view of Arithmetic, can answer the question proposed.

Ex. 6. A tradesman, in purchasing a piece of stuff, finds that if he had bought a yards more, at b pence a yard less, he would have paid the same sum: but if he had bought c yards more at d pence a yard less, his payment would have been e pence less: required the number of yards, and the price per yard.

Let x = the number of yards, and y = the price per yard:

$$\text{then, } xy = (x + a)(y - b), \text{ or } ay - bx = ab:$$

$$\text{also, } (x + c)(y - d) = xy - e, \text{ or } cy - dx = cd - e:$$

from (1), $acy - bcx = abc$:

from (2), $acy - adx = acd - ae$:

$\therefore (ad - bc)\,x = abc - acd + ae$, and $x = \dfrac{a\,(bc - cd + e)}{ad - bc}$:

whence, $y = \dfrac{b\,(ad - cd + e)}{ad - bc}$: and thus, both the number of yards, and the price per yard, are found.

In order that this solution may accord with the enunciation of the problem, it is manifest that $bc - cd + e$ and $ad - cd + e$ must be both positive, or both negative, according as ad is greater or less than bc.

(α) If the value of x be negative, whilst that of y is positive, and $-x$ be put for x, the two equations become

$$(-x + a)\,(y - b) = -xy, \text{ and } (-x + c)\,(y - d) = -xy - e,$$

which are respectively equivalent to

$$(x - a)\,(y - b) = xy, \text{ and } (x - c)\,(y - d) = xy + e:$$

and these translated into common language give the following problem, involving *inconsistent* conditions.

"A tradesman, in purchasing a piece of stuff, finds that if he had bought a yards *fewer* at b pence a yard *less*, he would have paid the same sum: but if he had bought c yards *fewer* at d pence a yard *less*, his payment would have been e pence *more*: required the number of yards, and the price per yard."

(β) If the value of x be positive, whilst that of y is negative, and $-y$ be substituted for y, the two equations become

$$(x + a)\,(-y - b) = -xy, \text{ and } (x + c)\,(-y - d) = -xy - e:$$

$$\text{or, } (x + a)\,(y + b) = xy, \text{ and } (x + c)\,(y + d) = xy + e:$$

to which corresponds the following problem similar to the last:

"A tradesman, in purchasing a piece of stuff, finds that if he had bought a yards *more*, at b pence a yard *more*, he would have paid the same sum: but if he had bought c yards

more, at d pence a yard *more*, his payment would have been e pence *more*: required the number of yards, and the price per yard."

(γ) If the values of x and y be both negative, and $-x$ and $-y$ be put in their places, the equations become

$$(-x+a)(-y-b)=(-x)(-y),$$

$$\text{and}\quad (-x+c)(-y-d)=(-x)(-y)-e:$$

$$\text{or}\quad (x-a)(y+b)=xy, \quad\text{and}\quad (x-c)(y+d)=xy-e,$$

which express the conditions of the following problem:

"A tradesman, in purchasing a piece of stuff, finds that if he had bought a yards *fewer*, at b pence a yard *more*, he would have paid the same sum: but if he had bought c yards *fewer*, at d pence a yard *more*, his payment would have been e pence *less*: required the number of yards, and the price per yard."

Finally, if it happen that $ad=bc$, or $ad-bc=0$: the reduced equations $ay-bx=ab$, and $cy-dx=cd-e$, become

$$ay-bx=ab, \text{ and } cy-\frac{bc}{a}x=cd-e, \because d=\frac{bc}{a}:$$

$$\therefore\ acy-bcx=abc, \text{ and } acy-bcx=acd-ae;$$

the former members of which being *identical*, independently of the values of x and y, the latter must manifestly be *equal* to each other: so that the number of independent equations is insufficient to render the problem determinate.

In this case, when $ad=bc$, and $abc=acd-ae$, we have

$$bc-cd+e=bc-c\frac{bc}{a}+e=\frac{1}{a}(abc-bc^2+ae):$$

$$\text{and}\quad ad-cd+e=a\frac{bc}{a}-c\frac{bc}{a}+e=\frac{1}{a}(abc-bc^2+ae):$$

$$\text{also,}\quad abc=acd-ae=ac\frac{bc}{a}-ae=bc^2-ae,$$

$$\text{or}\quad abc-bc^2+ae=0:$$

whence it follows that the values of x and y are expressed in the form $\frac{0}{0}$, from which no conclusion can be drawn: and in fact, this very form is a general indication of the *indeterminate* character of the question proposed, under such circumstances.

Ex. 7. *A*, *B*, *C* possess certain sums of money, such that if *A* receive half the sums of *B* and *C*, he will possess £a: if *B* receive a third of the sums of *A* and *C*, he will possess £b: and if *C* receive a fourth of the sums of *A* and *B*, he will be possessed of £c: required the original sums of each.

Let x, y, z denote the required sums: then will

$$x + \tfrac{1}{2}(y + z) = a, \quad \text{or } 2x + y + z = 2a:$$

$$y + \tfrac{1}{3}(x + z) = b, \quad \text{or } 3y + x + z = 3b:$$

$$z + \tfrac{1}{4}(x + y) = c, \quad \text{or } 4z + x + y = 4c:$$

from which we immediately find

$$x = \frac{1}{17}(22a - 9b - 8c):$$

$$y = \frac{1}{17}(21b - 4c - 6a):$$

$$z = \frac{1}{17}(20c - 3b - 4a).$$

The interpretation of the results obtained in the last example, was so fully dwelt upon, that it will suggest to the student how to modify the enunciation of this problem, so as to be accordant with the positive and negative values of x, y, z, which may arise from the relative magnitudes of the quantities a, b, c.

For some additional information upon the subjects of this chapter, see the first *Appendix:* and the student is further referred to *Bland's* Algebraical Problems.

CHAPTER VII.

RATIO, PROPORTION AND VARIATION.

RATIO.

162. Def. Ratio is the relation which subsists between two quantities of the same kind, the comparison being made by considering what multiple, part or parts, one of them is of the other: or, it is the arithmetical value of one of them, when the other is regarded as the unit.

Thus, if a and b be any two quantities whatever of the same kind, the ratio of a to b, will be represented by the fraction $\frac{a}{b}$, which indicates what multiple, part or parts, a is of b: or it is the Arithmetical value of a with reference to b considered as the unit.

The fraction $\frac{a}{b}$ is frequently written in the form $a : b$, in which a is termed the *Antecedent*, and b the *Consequent* of the ratio: also, $a : b$ is said to be a ratio of *Equality* when $a = b$: and to be a ratio of *Greater* or *Less Inequality*, according as a is greater or less than b.

163. Cor. 1. Hence, the magnitudes of ratios may be compared, by comparing the vulgar fractions which express their values.

Thus, the ratio of a to b is greater than, equal to, or less than the ratio of c to d, according as $\frac{a}{b}$ is greater than, equal to, or less than $\frac{c}{d}$: and therefore according as ad is greater than, equal to, or less than bc.

164. Cor. 2. Of two ratios, if the antecedents be equal, the ratio is the greater which has the less consequent: and if the consequents be equal, the ratio is the greater which has the greater antecedent.

165. *A ratio of greater inequality is diminished, and a ratio of less inequality is increased, by adding the same quantity to both its terms.*

Let $a : b$ be a ratio of inequality, and suppose x to be added to both its terms, so that it becomes $a + x : b + x$; then, the ratio $a : b$ is greater or less than the ratio $a + x : b + x$,

according as $\frac{a}{b}$ is greater or less than $\frac{a+x}{b+x}$,

according as $ab + ax$ is greater or less than $ab + bx$,

according as ax is greater or less than bx,

according as a is greater or less than b:

that is, if a be greater than b, the original ratio is greater than the new ratio, or the ratio has been diminished: and if a be less than b, the original ratio is less than the new ratio, or the ratio has been increased.

166. *A ratio of greater inequality is increased, and a ratio of less inequality is diminished, by subtracting a quantity less than either of them from both its terms.*

Let $a + x : b + x$ be the proposed ratio; and suppose x to be subtracted from both its terms so that it becomes $a : b$: then, $a + x : b + x$ is greater or less than $a : b$,

according as $\frac{a+x}{b+x}$ is greater or less than $\frac{a}{b}$,

according as $ab + bx$ is greater or less than $ab + ax$,

according as $bx + x^2$ is greater or less than $ax + x^2$,

according as $b + x$ is greater or less than $a + x$.

This proposition might have been established in the same manner as the last: and if the quantity subtracted

from the terms be greater than either of them, or the resulting ratio be of a symbolical form, the enunciated property no longer holds good.

167. Def. If the antecedents of two or more ratios be multiplied together for a new antecedent, and their consequents be multiplied together for a new consequent, the resulting ratio is said to be *compounded* of the others, and is sometimes termed their *sum*.

Thus, if $a : b$, $c : d$, $e : f$, &c. be any ratios: their compound ratio will be ace &c. : bdf &c.,

$$\text{or } = \frac{ace \text{ \&c.}}{bdf \text{ \&c.}}.$$

168. Cor. 1. If the antecedent of the succeeding ratio be always the consequent of the preceding one, so that $a : b$, $b : c$, $c : d$, &c. $x : y$ are the ratios; then the compound ratio will be

$$abc \text{ \&c. } x : bcd \text{ \&c. } y = \frac{abc \text{ \&c. } x}{bcd \text{ \&c. } y} = \frac{a}{y} = a : y:$$

or the ratio of the first antecedent to the last consequent.

169. Cor. 2. If the ratio $a : b$ be compounded with the ratio $x : y$, there results the ratio $ax : by = \frac{ax}{by}$, which is greater or less than $a : b = \frac{a}{b}$, according as x is greater or less than y.

If $x = y$, the ratio is not altered, as also appears from (1) of article (10).

170. Cor. 3. If there be m ratios each equal to $a : b$, the compound ratio will be

$$aaa \text{ \&c. to } m \text{ factors} : bbb \text{ \&c. to } m \text{ factors} = a^m : b^m:$$

and if m be assumed equal to 1, 2, 3, &c. in succession, the resulting ratio is styled the *simple*, *duplicate*, *triplicate*, &c. ratio of $a : b$; and sometimes, its *single*, *double*, *treble*, &c.

By an extension of this kind of notation and nomenclature, the ratios expressed by $a^{\frac{1}{2}} : b^{\frac{1}{2}}$, $a^{\frac{1}{3}} : b^{\frac{1}{3}}$, &c. are termed the *subduplicate*, *subtriplicate*, &c. ratio of $a : b$, and in some cases the *half*, *third*, &c. of it.

The ratio $a^{\frac{3}{2}} : b^{\frac{3}{2}}$ is called the *sesquiplicate* ratio of $a : b$.

171. COR. 4. Hence the indices, 2, 3, &c. m: $\frac{1}{2}$, $\frac{1}{3}$, &c. $\frac{1}{m}$, are sometimes called the *measures* of the corresponding ratios.

172. If $x : a$ be any ratio, wherein x is small compared to a: then, since

$$\frac{x}{a} = \frac{x^2}{ax} = \frac{x^3}{ax^2} = \&c.$$

it follows that x^2 is small compared with ax: x^3 compared with ax^2: and so on.

Also, since $\frac{x^2}{a^2} = \left(\frac{x}{a}\right)^2 = \frac{x}{a} \times \frac{x}{a}$, we have

$$\frac{x^2}{a^2} \div \frac{x}{a} = \frac{x}{a} \div 1, \text{ or } \frac{x^2}{a^2} : \frac{x}{a} = \frac{x}{a} : 1;$$

and because x is small compared with a, and therefore $\frac{x}{a}$ small compared with 1, it follows that $\frac{x^2}{a^2}$ is small compared with $\frac{x}{a}$, and *a fortiori*, much smaller compared with 1.

Similarly, it may be shewn that $\frac{x^3}{a^3}$ is small compared with $\frac{x^2}{a^2}$, and much more so compared with $\frac{x}{a}$ and 1: and so on.

In cases of this kind, the quantities $\frac{x}{a}$, $\frac{x^2}{a^2}$, $\frac{x^3}{a^3}$, &c. are sometimes termed *small* quantities of the *first*, *second*, *third*, &c. *orders*.

173. If the difference between the antecedent and consequent of a ratio be small compared to either of them, useful practical approximations to the ratios of their *squares* and *cubes*, may be readily obtained.

Let $a+x : a$ be a ratio, wherein x is very small compared to a; then we have

$$(a+x)^2 : a^2 = \frac{(a+x)^2}{a^2} = \frac{a^2+2ax+x^2}{a^2}$$

$$= 1 + 2\frac{x}{a} + \frac{x^2}{a^2}$$

$$= 1 + 2\frac{x}{a}, \text{ nearly, by the last article,}$$

$$= \frac{a+2x}{a}, \text{ nearly} = a+2x : a, \text{ nearly}:$$

that is, the ratio of the *squares* is nearly obtained by *doubling* the difference.

Again, $(a+x)^3 : a^3 = \dfrac{(a+x)^3}{a^3}$

$$= \frac{a^3+3a^2x+3ax^2+x^3}{a^3} = 1 + 3\frac{x}{a} + 3\frac{x^2}{a^2} + \frac{x^3}{a^3}$$

$$= 1 + 3\frac{x}{a}, \text{ nearly, by the last article,}$$

$$= \frac{a+3x}{a}, \text{ nearly} = a+3x : a, \text{ nearly}:$$

or, the ratio of the *cubes* is found nearly by *trebling* the difference.

Ex. The value of $(1002)^2 : (1000)^2$ is $1004 : 1000$ nearly: the actual ratio being $1004004 : 1000000 = 1004.004 : 1000$, by dividing its terms by 1000.

Similarly, $(7399)^3 : (7398)^3$ is very nearly expressed by $7402 : 7398$: and this is much more distinct and manageable than the true ratio, which would arise from performing the indicated operations.

In the same manner, good approximations to the square and cube roots, may be obtained by dividing the difference by 2 and 3 respectively.

PROPORTION.

174. Def. Proportion is the relation of equality subsisting between two ratios: thus,

if $a : b$ and $c : d$ be two ratios,

the equality $a : b = c : d$, or $\frac{a}{b} = \frac{c}{d}$ constitutes a proportion, which is frequently written $a : b :: c : d$; and read, as a is to b so is c to d.

The terms a, d are called the *extremes*, and the terms b, c the *means*.

175. Cor. 1. From the equality expressed by $\frac{a}{b} = \frac{c}{d}$, we have $\frac{a}{b} \times bd = \frac{c}{d} \times bd$, or $ad = bc$: that is, in every proportion, the product of the extremes is equal to the product of the means.

This equation enables us to find the value of any one of the terms, when the rest are given: thus,

$$a = \frac{bc}{d}, \quad b = \frac{ad}{c}, \quad c = \frac{ad}{b}, \quad d = \frac{bc}{a}.$$

Also, conversely the equality $ad = bc$, is easily made to assume the form of a proportion, whereof the extremes are the factors of one of its members, and the means those of the other: thus,

$$\text{since } \frac{ad}{bd} = \frac{bc}{bd}, \text{ we have } \frac{a}{b} = \frac{c}{d}:$$

$$\text{and therefore } a : b = c : d.$$

176. Cor. 2. If three quantities a, b, c form what is called a *continued* proportion, so that $a : b = b : c$: we shall have $ac = b^2$; or the product of the extremes is equal to the square of the mean.

177. From what has been already said, it appears that the doctrine of Proportion, is merely the determination of the relations of fractions, whose numerators are the antecedents, and denominators the consequents of the ratios which constitute them: therefore, of the four quantities a, b, c, d which form a Proportion, there may be made various other arrangements and modifications, in which proportionality will still be preserved, and of these the most useful are the following.

(1) Since $\frac{a}{b} = \frac{c}{d}$, $\therefore \frac{a}{b} \times \frac{b}{c} = \frac{c}{d} \times \frac{b}{c}$, or $\frac{a}{c} = \frac{b}{d}$:

that is, $a : c = b : d$. (*Alternando.*)

(2) Since $\frac{a}{b} = \frac{c}{d}$, $\therefore 1 \div \frac{a}{b} = 1 \div \frac{c}{d}$, or $\frac{b}{a} = \frac{d}{c}$:

that is, $b : a = d : c$. (*Invertendo.*)

(3) Since $\frac{a}{b} = \frac{c}{d}$, $\therefore \frac{a}{b} + 1 = \frac{c}{d} + 1$, or $\frac{a+b}{b} = \frac{c+d}{d}$:

that is, $a + b : b = c + d : d$. (*Componendo.*)

(4) Since $\frac{a}{b} = \frac{c}{d}$, $\therefore \frac{a}{b} - 1 = \frac{c}{d} - 1$, or $\frac{a-b}{b} = \frac{c-d}{d}$:

that is, $a - b : b = c - d : d$. (*Dividendo.*)

(5) Since $\frac{a-b}{b} = \frac{c-d}{d}$ and $\frac{b}{a} = \frac{d}{c}$, $\therefore \frac{a-b}{a} = \frac{c-d}{c}$:

that is, $a - b : a = c - d : c$. (*Convertendo.*)

(6) Since $\frac{a+b}{b} = \frac{c+d}{d}$ and $\frac{a-b}{b} = \frac{c-d}{d}$, $\therefore \frac{a+b}{a-b} = \frac{c+d}{c-d}$:

that is, $a+b : a-b = c+d : c-d$. (*Componendo & Dividendo.*)

(7) Since $\frac{a}{b} = \frac{c}{d}$, we have $\frac{ma}{mb} = \frac{nc}{nd}$: whence we obtain $ma : mb = nc : nd$, where m and n are either integral or fractional, or indeed general symbols.

(8) Since $\frac{a}{b} = \frac{c}{d}$, we have $\frac{ma}{nb} = \frac{mc}{nd}$: from which we obtain $ma : nb = mc : nd$, where m and n may be either integral or fractional, or general symbols.

(9) Since $\frac{a}{b} = \frac{c}{d}$, we have $\left(\frac{a}{b}\right)^m = \left(\frac{c}{d}\right)^m$, or $\frac{a^m}{b^m} = \frac{c^m}{d^m}$: whence, $a^m : b^m = c^m : d^m$, where m may be of any form whatever.

178. Cor. Since $\frac{a-b}{a} = \frac{c-d}{c}$, or $\frac{a-b}{c-d} = \frac{a}{c}$: if a be the greatest term of the proportion, and consequently d the least, we shall have

$$a - b > c - d:$$

$$\text{also, } b + d = b + d:$$

whence, $a + d > b + c$: or the greatest and least terms together, are greater than the other two together.

179. Similar considerations lead to the determination of the relations subsisting between different proportions, as will appear in the following instances.

(1) If $a : b = c : d$, and $c : d = e : f$, be two proportions:

then, since $\frac{a}{b} = \frac{c}{d}$ and $\frac{c}{d} = \frac{e}{f}$: we have

$\frac{a}{b} = \frac{e}{f}$, or $a : b = e : f$: similarly of more.

(2) If $a : b = c : d$, and $e : b = f : d$:

then, $\frac{a}{b} = \frac{c}{d}$, and $\frac{e}{b} = \frac{f}{d}$:

whence, $\frac{a \pm e}{b} = \frac{c \pm f}{d}$: or $a \pm e : b = c \mp f : d$.

(3) If any number of magnitudes a, b, c, d, e, f, &c. be so circumstanced, that $a : b = c : d = e : f =$ &c.:

$$\text{then, } \frac{a}{b} = \frac{a}{b}, \quad \frac{a}{b} = \frac{c}{d}, \quad \frac{a}{b} = \frac{e}{f}, \text{ \&c.}:$$

$$\therefore\ ab = ba, \quad ad = bc, \quad af = be, \text{ \&c.}.$$

$$\therefore\ ab + ad + af + \text{\&c.} = ba + bc + be + \text{\&c.}:$$

$$\text{or, } a(b + d + f + \text{\&c.}) = b(a + c + e + \text{\&c.}):$$

$$\text{whence, } \frac{a}{b} = \frac{a + c + e + \text{\&c.}}{b + d + f + \text{\&c.}}:$$

that is, as one antecedent is to its consequent, so is the sum of all the antecedents to the sum of all the consequents.

Similarly, if $a : b = b : c = c : d =$ &c., we shall have

$$\frac{a}{b} = \frac{a + b + c + \text{\&c.}}{b + c + d + \text{\&c.}}.$$

(4) From $\frac{a}{b} = \frac{c}{d}$, and $\frac{e}{f} = \frac{g}{h}$, we have

$$\frac{a}{b} \times \frac{e}{f} = \frac{c}{d} \times \frac{g}{h}, \text{ or } \frac{ae}{bf} = \frac{cg}{dh}:$$

and the proportion $ae : bf = cg : dh$, is said to be *compounded* of the two proportions,

$$a : b = c : d, \text{ and } e : f = g : h,$$

by multiplying together their corresponding terms: also, the same holds good, whatever be the number of *component* proportions.

Most of the results contained in the last three articles are of great practical utility, and are frequently enunciated at length, so as to assume the form of *Rules*.

180. If three magnitudes a, b, c be in continued proportion, so that $\frac{a}{b} = \frac{b}{c}$: then, we shall have

$$\frac{a}{c} = \frac{a}{b} \times \frac{b}{c} = \frac{a}{b} \times \frac{a}{b} = \frac{a^2}{b^2}:$$

$$\text{that is, } a : c = a^2 : b^2:$$

or, the first has to the third, the same ratio as the square of the first has to the square of the second.

Again, if four magnitudes be in continued proportion, so that $\frac{a}{b} = \frac{b}{c} = \frac{c}{d}$: then will

$$\frac{a}{d} = \frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} = \frac{a^3}{b^3}:$$

$$\text{that is, } a : d = a^3 : b^3:$$

or, the first has to the fourth, the same ratio as the cube of the first has to the cube of the second.

The two results here obtained include the *geometrical* definitions of duplicate and triplicate ratio.

Similarly, if there be m quantities a, b, c, &c. l circumstanced as in the preceding cases: it will readily appear that

$$a : l = a^{m-1} : b^{m-1}.$$

181. In (8) of article (177), we have seen that

$$\frac{ma}{nb} = \frac{mc}{nd}:$$

wherefore, if ma be greater than, equal to, or less than nb, it follows that mc will be greater than, equal to, or less than nd.

Hence, of the terms of a proportion there being taken any equimultiples *whatever* of the first and third, and any equimultiples *whatever* of the second and fourth: if the multiple of the first be greater than the multiple of the second, the multiple of the third will be greater than that of the fourth: if equal, equal, and if less, less.

Also conversely, if a, b, c, d be four magnitudes so circumstanced that ma is always greater than, equal to, or less than nb, according as mc is greater than, equal to, or less than nd, whatever whole numbers m and n may be: then will a, b, c, d taken in order, be the terms of a proportion.

For, if not, let a, b, c, e form a proportion, so that

$$\frac{a}{b} = \frac{c}{e}, \text{ and } \therefore \frac{ma}{nb} = \frac{mc}{ne}:$$

then ma will be greater than, equal to, or less than nb, according as mc is greater than, equal to, or less than ne, for all values of m and n: that is, ne and nd possess the same properties, and must therefore be equal to each other, or $e = d$: and consequently a, b, c, d are the terms of a proportion.

182. Cor. The characteristic of proportionality which has just been deduced from the arithmetical or algebraical view of the subject, is manifestly applicable to all kinds of magnitudes whatever, inasmuch as it will always be possible to *repeat* them as often as we please: and it is found to agree with the *Geometrical* definition of proportion as laid down in the fifth definition of the fifth book of *Euclid's* Elements.

Since the principles of *Geometry* furnish us with no means of expressing the value of the fraction $\frac{a}{b}$, where a and b are geometrical magnitudes, it is evident that we can have no geometrical definition of Ratio at all in accordance with the view of article (162): and consequently the definition of proportion adopted in article (174) will at once become useless, when applied to magnitudes of this description. But, from the last article, it appears that the algebraical definition of proportion, as above considered, includes also the geometrical definition, so that whenever a proportion exists among four arithmetical magnitudes, the same will subsist among four geometrical magnitudes, of which they may have been assumed to be the numerical representatives.

Many geometrical magnitudes being arithmetically represented by surds or incommensurable quantities, it would be impossible to ascertain what multiple, part or parts one may be of another: that is, number being a *discrete* and extension a *continuous* magnitude, it is evident that though the parts of number are always easily and distinctly assigned, the parts

of extension incapable of exact numerical representation, can not be determined in the same manner: and accordingly, recourse is had to *multiples* instead of *aliquot parts:* and the geometrical definition of proportion is a necessary consequence of that adopted in Arithmetic and Algebra.

183. We will conclude this subdivision with the following additional definitions, and some consequences resulting immediately from them.

(1) Three quantities a, b, c are in *Arithmetical proportion*, when $\dfrac{a-b}{b-c}=1$, or $a-b=b-c$.

(2) Three quantities a, b, c are in *Geometrical proportion*, when $\dfrac{a}{b}=\dfrac{b}{c}$, or $a:b=b:c$.

(3) Three quantities a, b, c are in *Harmonical proportion*, when $\dfrac{a}{c}=\dfrac{a-b}{b-c}$, or $a:c=a-b:b-c$.

184. COR. Hence these three proportions are connected with each other by the following forms.

$a:a=a-b:b-c$, for Arithmetical Proportion:

$a:b=a-b:b-c$, for Geometrical Proportion:

$a:c=a-b:b-c$, for Harmonical Proportion:

in which the consequents of the former ratios are the three quantities a, b c, and all the other terms are the same.

185. To find an arithmetic, geometric and harmonic mean between two quantities a and b.

Let $x=$ the arithmetic mean, so that a, x, b are in arithmetical proportion: then,

$a-x=x-b$, and $x=\frac{1}{2}(a+b)$, the arithmetic mean:

let $y=$ the geometric mean, so that $a:y=y:b$;

then, $ab=y^2$, and $y=\sqrt{ab}$, the geometric mean:

let $z =$ the harmonic mean, so that $a : b = a - z : z - b$;

then, $az - ab = ab - bz$, and $z = \frac{2ab}{a+b}$, the harmonic mean.

186. Cor. From these results, we obtain

$$xz = \tfrac{1}{2}(a+b) \times \frac{2ab}{a+b} = ab = y^2:$$

$$\therefore \frac{x}{y} = \frac{y}{z}, \text{ or } x : y = y : z;$$

that is, the three means are in continued proportion, as might have been expected from the forms exhibited in the last article.

Also,

since $\frac{1}{2}(a+b) - \sqrt{ab} = \frac{1}{2}(a - 2\sqrt{ab} + b) = \frac{1}{2}(\sqrt{a} - \sqrt{b})^2$ is a *positive* quantity, when a is not equal to b, it follows that x is greater than y, and consequently y greater than z.

VARIATION.

187. Def. A quantity is said to *vary* as one or more others, when it is so dependent upon them, that every change which they undergo, produces a corresponding and *proportional* change in its magnitude: and it is consequently connected with them, by some multiplier either integral or fractional, which remains the same in the whole of any operation wherein they are concerned.

The different kinds of Variation are distinguished as follows, the sign $\propto$ expressing this connection, and p being a factor which undergoes no change however the magnitudes denoted by A, B, C may vary.

(1) If $A = pB$, A varies *directly* as B, or $A \propto B$:

(2) If $A = \frac{p}{B}$, A varies *inversely* as B, or $A \propto \frac{1}{B}$:

(3) If $A = pBC$, A varies as B and C *jointly*, or $A \propto BC$:

(4) If $A = p\frac{B}{C}$, A varies as B *directly* and C *inversely*,

$$\text{or } A \propto \frac{B}{C}:$$

and the same may be extended to any number of magnitudes whatever.

It is obvious that the Variation here intended, is merely an abbreviation of the method of expressing proportions: thus, if

$$A : B = a : b, \text{ we have } A = \frac{a}{b}B,$$

in which $\frac{a}{b}$ may be replaced by the invariable quantity p above used:

again, if $A : \frac{1}{B} = a : \frac{1}{b}$, we have $A = \frac{ab}{B}$, whereof ab may be represented by the symbol p as before: and similarly of the rest.

188. The fundamental rules as laid down in the preceding pages, will lead immediately to all the consequences which the view of variation above adopted presents.

(1) *If* $\mathrm{A} \propto \mathrm{B}$ *and* $\mathrm{B} \propto \mathrm{C}$: *then will* $\mathrm{A} \propto \mathrm{C}$.

$$\text{For, if } A = pB \text{ and } B = qC,$$

$$\text{we have } A = pB = pqC; \text{ that is, } A \propto C.$$

Hence also, if $A \propto \frac{1}{B}$ and $B \propto C$, we have

$$A = \frac{p}{B}, \; B = qC, \text{ and } \therefore\; A = \frac{p}{q}\frac{1}{C}, \text{ or } A \propto \frac{1}{C}.$$

(2) *If* $\mathrm{A} \propto \frac{1}{\mathrm{B}}$ *and* $\mathrm{B} \propto \frac{1}{\mathrm{C}}$: *then will* $\mathrm{A} \propto \mathrm{C}$.

For, if $A = \frac{p}{B}$ and $B = \frac{q}{C}$, then $A = \frac{p}{q}C$, or $A \propto C$.

In the same manner, whatever be the number of magnitudes, when each varies inversely as the following, the first will vary *directly* or *inversely* as the last, according as the number of intermediate magnitudes is *odd* or *even*.

(3) *If* $A \propto C$ *and* $B \propto C$; *then will* $A \pm B \propto C$, *and* $\sqrt{AB} \propto C$.

For,

if $A = pC$ and $B = qC$, then $A \pm B = (p \pm q)C$, or $A \pm B \propto C$:

also, $AB = pqC^2$, and $\therefore \sqrt{AB} = \sqrt{pq}C$, or $\sqrt{AB} \propto C$:

and similar conclusions may be drawn whatever be the number of quantities employed.

(4) *If* $A \propto B$, *then will* $AP \propto BP$, *and* $\frac{A}{P} \propto \frac{B}{P}$, *where* P *may be either variable or invariable.*

For, if $A = pB$, we have $AP = pBP$, and $\frac{A}{P} = p\frac{B}{P}$:

whence it follows that $AP \propto BP$, and $\frac{A}{P} \propto \frac{B}{P}$.

Hence also, $A^m = p^m B^m$ and $A^m \propto B^m$, where m may be either integral or fractional.

(5) *If* $A \propto BC$, *then will* $B \propto \frac{A}{C}$, *and* $C \propto \frac{A}{B}$.

For, if $A = pBC$, then $B = \frac{1}{p}\frac{A}{C} \propto \frac{A}{C}$, and $C = \frac{1}{p}\frac{A}{B} \propto \frac{A}{B}$.

Hence also, if A be invariable and equal to q, we have

$$B = \frac{q}{p}\frac{1}{C} \propto \frac{1}{C}, \text{ and } C = \frac{q}{p}\frac{1}{B} \propto \frac{1}{B}.$$

(6) *If* $A \propto B$ *and* $C \propto D$, *then will* $AC \propto BD$, *and* $\frac{A}{C} \propto \frac{B}{D}$.

For, if $A = pB$ and $C = qD$, then $AC = pqBD \propto BD$:

$$\text{also, } \frac{A}{C} = \frac{p}{q}\frac{B}{D} \propto \frac{B}{D}.$$

Similar results will be obtained whatever be the number of quantities concerned.

(7) *If* A $\propto$ B *when* C *is invariable, and* A $\propto$ C *when* B *is invariable, then will* A $\propto$ BC, *when both* B *and* C *are variable.*

For, we may manifestly assume $A = pCB$, and $A = qBC$: whence, $A^2 = pq\,(BC)^2$, or $A \propto BC$.

(8) From the proportion $A : B = C : D$, we have

$$A = \frac{BC}{D}: \text{ whence } A \propto BC, \text{ when } D \text{ is given:}$$

$A \propto \dfrac{B}{D}$, when C is given, and $A \propto \dfrac{1}{D}$, when B and C are given.

(9) *When the change in a quantity depends upon the changes in several others, and it appears that the former quantity is always proportional to each of the latter when the rest are invariable, then shall the former vary as the continued product of the latter, when they are all variable.*

Let A vary as each of the quantities B, C, D, &c. L when the rest are invariable: and instead of the changes taking place *simultaneously*, let them take place *separately*, and be such that when B is changed to b, A is changed to A_1: when C is changed to c, A_1 is changed to A_2: when D is changed to d, A_2 is changed to A_3, &c.: and finally when L is changed to l, A_n is changed to a: then we shall have the following proportions:

$$A : A_1 = B : b;$$
$$A_1 : A_2 = C : c;$$
$$A_2 : A_3 = D : d;$$
$$\&c. = \&c.$$
$$A_n : a = L : l;$$

and these being compounded, and the common factors rejected from the terms of the former resulting ratio, we shall have

$$A : a = BCD \;\&c.\; L : bcd \;\&c.\; l;$$

$$\text{and therefore } A \propto BCD \;\&c.\; L.$$

This comprises the purport of (7) established by a somewhat different process, though dependent upon the same principles.

The proportion $A : a = BCD$ &c. $L : bcd$ &c. l, includes the Arithmetical Rules of *simple* and *compound Proportion*, where A and a are the *effects* produced, and B, C, D, &c. L, b, c, d, &c. l, the *agents* employed for that purpose.

For examples, the student is referred to articles (130) and (131), of the Author's *Arithmetic*.

We will conclude this chapter with the application of its principles to a few simple instances.

(1) To find two magnitudes in the ratio of $m : n$, such that if a be added to each, the sums shall be in the ratio of $p : q$.

Let mx and nx, having the given ratio, denote the quantities required: then,

$$\frac{mx + a}{nx + a} = \frac{p}{q}, \text{ by the question:}$$

$$\text{from which we obtain } x = \frac{(p - q)a}{mq - np}:$$

$$\text{and } \therefore \frac{(p-q)ma}{mq - np} \text{ and } \frac{(p - q)na}{mq - np},$$

will answer the conditions of the problem.

(2) Required two numbers in the ratio of $4 : 5$, from which, if two other required numbers in the ratio of $6 : 7$, be respectively subtracted, the remainders shall be in the ratio of $2 : 3$, and their sum equal to 20.

Let $4x$ and $5x$ denote the first two numbers, and $6y$ and $7y$ the other two:

$$\text{then, } \frac{4x - 6y}{5x - 7y} = \frac{2}{3}, \text{ and } 9x - 13y = 20, \text{ by the question:}$$

whence, $y = 4$ and $x = 8$: so that the first two numbers are 32 and 40, and the other two 24 and 28.

(3) Of two vessels one contains water and the other brandy: half the water is poured into the brandy, and an equal quantity of the mixture is poured back into the water, when the vessel is found to contain m times as much water as brandy; compare the water and the brandy.

Let x and y denote the quantities of water and brandy respectively: then

$$y + \tfrac{1}{2}x \text{ of mixture contains } y \text{ of brandy}:$$

$$\therefore\ y + \tfrac{1}{2}x : \tfrac{1}{2}x :: y : \frac{xy}{2y+x} = \text{brandy poured back}:$$

$$\text{whence, } \frac{xy}{2y+x} = \frac{1}{m}\left\{x - \frac{xy}{2y+x}\right\}, \text{ by the question}:$$

$$\text{that is, } mxy = xy + x^2, \text{ or } (m-1)y = x:$$

$$\text{and } \therefore\ x : y = m - 1 : 1, \text{ the required ratio.}$$

(4) Resolve the number 24 into two factors, so that the sum of their cubes may be to the difference of their cubes as $35 : 19$.

Let x and y denote the required factors: then, $xy = 24$, and $x^3 + y^3 : x^3 - y^3 = 35 : 19$:

$\therefore$ *componendo* and *dividendo*, we have

$$2x^3 : 2y^3 = 54 : 16, \text{ or } x^3 : y^3 = 27 : 8:$$

$$\text{whence, } 8x^3 = 27y^3 = 27\left(\frac{24}{x}\right)^3, \text{ or } 8x^6 = 27(24)^3:$$

$$\therefore\ 2x^2 = 3 \times 24 = 72,\ x^2 = 36, \text{ and } x = \pm 6:$$

$$\therefore\ y = \frac{24}{x} = \frac{24}{\pm 6} = \pm 4:$$

that is, 6 and 4 are the factors required: the values -6 and -4 of x and y giving merely a symbolical solution.

(5) If $y = p + q + r$, where p is invariable, q varies as x, and r varies as x^2; find the relation between y and x, supposing that when $x = 1$, $y = 6$: when $x = 2$, $y = 11$: and when $x = 3$, $y = 18$.

Let $y = a + \beta x + \gamma x^2$, be the required relation: then, by the conditions of the question, we have given

$$6 = a + \beta + \gamma:$$

$$11 = a + 2\beta + 4\gamma:$$

$$18 = a + 3\beta + 9\gamma:$$

to find the values of a, β and γ:

$$\text{from (1) and (2)}, \quad 5 = \beta + 3\gamma: \qquad (4)$$

$$\text{from (1) and (3)}, \quad 12 = 2\beta + 8\gamma: \qquad (5)$$

$$\text{from (4)}, \quad 10 = 2\beta + 6\gamma:$$

$$\text{from (5)}, \quad 12 = 2\beta + 8\gamma:$$

whence, $2\gamma = 2$ and $\gamma = 1$; $\therefore \beta = 5 - 3\gamma = 5 - 3 = 2$:

and $a = 6 - \beta - \gamma = 6 - 2 - 1 = 3$:

that is, $y = 3 + 2x + x^2$, is the relation sought.

For additional problems upon these subjects, see the Appendices at the end of the work.

193. COR. 3. In an arithmetical progression, the sum of any two terms, equidistant from the extremes, is always equal to the sum of the extremes.

For, if a and l be the first and last terms:

then, the p^{th} term from the beginning $= a + (p - 1) d$:

and the p^{th} term from the end $= l - (p - 1) d$:

whence, the sum of these $= a + l$

$=$ the sum of the extremes.

Ex. 1. Find the n^{th} term, and also the sum of n terms of the series of odd natural numbers, 1, 3, 5, 7, &c.

Generally, $l = a + (n - 1) d$, and $s = \{2a + (n - 1) d\} \frac{n}{2}$:

and here $a = 1$, $d = 2$; whence, by substitution, we find

$$l = 1 + (n - 1) 2 = 2n - 1:$$

$$s = \{2 + (n - 1) 2\} \frac{n}{2} = n^2:$$

that is, the n^{th} of the odd natural numbers beginning with 1, is expressed by $2n - 1$, and the sum of the first n of them by n^2: as may easily be verified.

Ex. 2. Find the n^{th} term, and the sum of n terms of the series of even natural numbers, 2, 4, 6, 8, &c.

Here, $a = 2$, and $d = 2$: and the formulæ give

$$l = 2n, \text{ and } s = n (n + 1):$$

that is, the n^{th} even natural number is expressed by $2n$, and the sum of the first n such numbers by $n (n + 1)$.

Ex. 3. Required the n^{th} term, and the sum of n terms of the series, $(b + x)^2$, $b^2 + x^2$, $(b - x)^2$, &c.

Here, $a = (b + x)^2$, and $d = (b^2 + x^2) - (b + x)^2 = -2bx$:

$$\text{whence, } l = (b + x)^2 - (n - 1) 2bx$$

$$= b^2 + x^2 - 2 (n - 2) bx:$$

$$s = \left\{2(b+x)^2 - (n-1)\,2bx\right\}\frac{n}{2}$$

$$= n(b^2 + x^2) - n(n-3)\,bx.$$

194. By means of the two fundamental equations:

(1), $\quad l = a + (n-1)\,d:$

(2), $\quad s = \left\{2a + (n-1)\,d\right\}\dfrac{n}{2}:$

if any three of the quantities involved be given, the remaining one may be found by the solution of the equations with respect to it: and from the two equations combined, it will not be difficult to arrive at the following results.

$$(\alpha)\quad a = l - (n-1)\,d = \frac{2s}{n} - l = \frac{s}{n} - \frac{(n-1)\,d}{2}$$

$$= \frac{d}{2} \pm \sqrt{\left(l + \frac{d}{2}\right)^2 - 2sd}.$$

$$(\beta)\quad l = a + (n-1)\,d = \frac{2s}{n} - a = \frac{s}{n} + \frac{(n-1)d}{2}$$

$$= -\frac{d}{2} \pm \sqrt{\left(a - \frac{d}{2}\right)^2 + 2sd}.$$

$$(\gamma)\quad d = \frac{l-a}{n-1} = \frac{2(s-na)}{n(n-1)} = \frac{2(nl-s)}{n(n-1)} = \frac{l^2-a^2}{2s-a-l}.$$

$$(\delta)\quad n = \frac{l-a+d}{d} = \frac{2s}{a+l} = \frac{1}{2} - \frac{a}{d} \pm \sqrt{\left(\frac{a}{d} - \frac{1}{2}\right)^2 + \frac{2s}{d}}$$

$$= \frac{1}{2} + \frac{l}{d} \pm \sqrt{\left(\frac{l}{d} + \frac{1}{2}\right)^2 - \frac{2s}{d}}.$$

$$(\epsilon)\quad s = \left\{2l - (n-1)\,d\right\}\frac{n}{2} = na + \frac{n(n-1)\,d}{2}$$

$$= nl - \frac{n(n-1)\,d}{2} = \frac{l+a}{2} + \frac{l^2-a^2}{2d}.$$

Ex. 1. Given $s = 72$, $a = 17$ and $d = -2$: to find the series.

From the general formula (1), we have immediately,

$$72 = \{34 - (n-1)2\}\frac{n}{2}:$$

$$\therefore\ 72 = 18n - n^2, \quad \text{or} \quad n^2 - 18n = -72:$$

and this solved, gives $n = 6$, and $n = 12$.

whence, the series corresponding will be

$$17,\ 15,\ 13,\ 11,\ 9,\ 7:$$

$$17,\ 15,\ 13,\ 11,\ 9,\ 7,\ 5,\ 3,\ 1,\ -1,\ -3,\ -5.$$

The first of these series is the direct arithmetical answer to the question proposed: but the second arises from the qualities assumed to be attached to the symbols by means of their algebraical signs, the last six terms of which having no arithmetical effect upon any of the quantities which are given.

Ex. 2. Given $s = 143$, $d = 2$ and $n = 11$: to find the series.

Here, $143 = \{2a + (11-1)2\}\dfrac{11}{2} = 11a + 110:$

$$\therefore\ 11a = 143 - 110 = 33, \text{ and } a = 3:$$

whence, the series is

$$3,\ 5,\ 7,\ 9,\ 11,\ 13,\ 15,\ 17,\ 19,\ 21,\ 23:$$

which will be found to satisfy the condition.

Ex. 3. Given $s = 136$, $a = 31$ and $d = -4$: to find the series.

Here, $136 = \{62 - (n-1)4\}\dfrac{n}{2} = 33n - 2n^2:$

and $l = 31 - (n-1)4 = 35 - 4n:$

from (1), $2n$ [illegible] $= -136$: from (2), $n = \dfrac{35 - l}{4}$:

whence, by substitution, we obtain

$$2\left(\frac{35-l}{4}\right)^2 - 33\left(\frac{35-l}{4}\right) = -136:$$

which, after the proper reductions, becomes $l^2 - 4l = -3$:

$$\therefore\ l^2 - 4l + 4 = -3 + 4 = 1:$$

$$\text{and } l - 2 = \pm 1, \text{ or } l = 3 \text{ and } 1.$$

Since when $l = 3$, $n = 8$, and when $l = 1$, $n = 8\frac{1}{2}$, it is evident that the latter value of l is not accordant with the nature of the case, because n is necessarily a whole number: and therefore the required series is

$$31,\ 27,\ 23,\ 19,\ 15,\ 11,\ 7,\ 3.$$

195. *To insert* m *arithmetic means between* a *and* b.

Let d be the common difference: then, since the number of terms = the number of means + the number of extremes, we shall have $n = m + 2$, using the preceding notation:

$$\therefore\ b = a + (n-1)\,d = a + (m+1)\,d:$$

whence, the common difference $d = \dfrac{b-a}{m+1}$, is found:

and the successive means are thence immediately obtained.

Ex. Find two arithmetic means between -3 and 3.

Here, $a = -3$, $b = 3$, and $m = 2$:

from which, by the formula, we have $d = \dfrac{3+3}{3} = 2$:

$$\therefore \text{ the first mean} = -3 + 2 = -1:$$

$$\text{the second mean} = -1 + 2 = +1:$$

and it is evident that -3, -1, $+1$ and $+3$, form an arithmetical progression in symbolical algebra.

196. COR. This proposition amounts to finding the arithmetical series, when the two extremes and the number of terms are given.

Thus, since $d = \dfrac{b - a}{n - 1}$, the series will be

$$a, \quad \frac{(n-2)\,a+b}{n-1}, \quad \frac{(n-3)\,a+2b}{n-1}, \text{ \&c.,}$$
$$\frac{2a+(n-3)\,b}{n-1}, \quad \frac{a+(n-2)\,b}{n-1}, \quad b.$$

Also, the m arithmetic means, inserted between a and b, will be expressed generally by

$$\frac{ma+b}{m+1}, \quad \frac{(m-1)a+2b}{m+1}, \text{ \&c.,} \quad \frac{2a+(m-1)\,b}{m+1}, \quad \frac{a+mb}{m+1}:$$

the symbols a and b being similarly employed in the terms reckoned from the beginning and the end, as they manifestly ought to be.

GEOMETRICAL PROGRESSION.

197. Def. A Geometrical Progression is a series of quantities in continued geometrical proportion, and therefore increasing or decreasing throughout by a *Common Ratio* or Factor.

$$\text{Thus,} \qquad a, \; 2a, \; 4a, \; 8a, \text{ \&c.}$$
$$a, \; ab, \; ab^2, \; ab^3, \text{ \&c.}$$
$$ax, \; -\frac{a^2}{x}, \; \frac{a^3}{x^3}, \; -\frac{a^4}{x^5}, \text{ \&c.}$$

are all geometrical progressions, the first and second *increasing* by the common ratios 2, and b respectively, and the third *decreasing* by the common ratio $-ax^{-2}$.

198. *In a geometrical progression, given the first term and the common ratio, to find the* n[th] *term: and also the sum of* n *terms.*

Let a denote the first term, r the common ratio, l the n^{th} term, and s the sum of n terms: then, the series will be

$$a, \; ar, \; ar^2, \; ar^3, \text{ \&c.}:$$

and since r is not found in the first term, and its index increases by 1 in each term from the second, we shall have

$$l = ar^{n-1}.$$

Also, $s = a + ar + ar^2 + \&c. + ar^{n-2} + ar^{n-1}$:

$\therefore\ rs = ar + ar^2 + ar^3 + \&c. + ar^{n-1} + ar^n$:

whence, subtracting the former from the latter, we obtain

$$(r-1)s = ar^n - a = a(r^n - 1):$$

$$\text{and } \therefore\ s = \frac{a(r^n - 1)}{r - 1}.$$

To these formulæ, the remarks made in article (190), are immediately applicable.

199. Cor. 1. Hence it follows that the terms of a geometrical progression, taken at equal intervals, are also in geometrical progression.

200. Cor. 2. The terms of the series, taken in the reverse order, will evidently be

$$l,\ \frac{l}{r},\ \frac{l}{r^2},\ \frac{l}{r^3},\ \&c.,\ \frac{l}{r^{n-1}}:$$

$$\therefore\ s = l + \frac{l}{r} + \frac{l}{r^2} + \frac{l}{r^3} + \&c. + \frac{l}{r^{n-1}}$$

$$= l + \frac{1}{r}\left(l + \frac{l}{r} + \frac{l}{r^2} + \&c. + \frac{l_{\prime}}{r^{n-2}}\right)$$

$$= l + \frac{1}{r}\left(s - \frac{l}{r^{n-1}}\right) = l + \frac{1}{r}(s - a):$$

$$\text{whence, } rs = rl + s - a, \text{ and } \therefore\ s = \frac{rl - a}{r - 1}:$$

which formula gives a practical rule.

201. Cor. 3. In a geometrical progression, the product of any two terms, equidistant from the extremes, is always equal to the product of the extremes.

For, if a and l be the first and last terms:

then, the p^{th} term from the beginning $= ar^{p-1}$:

and, the p^{th} term from the end $= \dfrac{l}{r^{p-1}}$:

whence, their product $= al =$ the product of the extremes.

Ex. 1. Required the n^{th} term, and the sum of n terms of the progression, 1, 2, 4, 8, &c.

Generally, $l = ar^{n-1}$, and $s = \dfrac{a(r^n - 1)}{r - 1}$:

and here, $a = 1$, $r = 2$: $\therefore$ $l = 2^{n-1}$, and $s = 2^n - 1$:

which may easily be verified for any particular value of n.

Ex. 2. Find the n^{th} term, and the sum of n terms of the series, 8, 20, 50, 125, &c.

Here, $a = 8$, $r = \dfrac{20}{8} = \dfrac{5}{2}$: and the formulæ give

$$l = 8 \times \left(\frac{5}{2}\right)^{n-1} = \frac{5^{n-1}}{2^{n-4}}: \quad s = \frac{8\left\{\left(\frac{5}{2}\right)^n - 1\right\}}{\frac{5}{2} - 1} = \frac{1}{3}\left\{\frac{5^n - 2^n}{2^{n-4}}\right\}.$$

Ex. 3. Determine the n^{th} term, and the sum of n terms of

$$\frac{1}{5}, \quad -\frac{2}{15}, \quad \frac{4}{45}, \quad - \text{\&c.}$$

Here, $a = \dfrac{1}{5}$, $r = -\dfrac{2}{15} \div \dfrac{1}{5} = -\dfrac{2}{3}$: and we have

$$l = \pm\frac{1}{5}\left(\frac{2^{n-1}}{3^{n-1}}\right), \quad s = \frac{1}{25}\left(\frac{3^n \pm 2^n}{3^{n-1}}\right):$$

where the upper or lower sign is applicable, according as n is odd or even.

202. By means of two fundamental equations,

$$(1) \quad l = ar^{n-1}: \qquad (2) \quad s = \frac{a(r^n - 1)}{r - 1}:$$

we shall readily arrive at the following results:

$$(\alpha) \quad a = \frac{l}{r^{n-1}} = \frac{(r-1)s}{r^n - 1} = rl - (r-1)s:$$

$$(\beta) \quad l = \frac{(r-1)r^{n-1}s}{r^n - 1} = s - \frac{s - a}{r}:$$

$$(\gamma) \quad r = \left(\frac{l}{a}\right)^{\frac{1}{n-1}} = \frac{s-a}{s-l}:$$

$$(\delta) \quad s = \frac{(r^n - 1)\,l}{r^{n-1}(r-1)} = \frac{l^{\frac{n}{n-1}} - a^{\frac{n}{n-1}}}{l^{\frac{1}{n-1}} - a^{\frac{1}{n-1}}}.$$

The values of n cannot be exhibited in terms of the rest, without the aid of logarithms: but in addition to these, we may easily deduce the following formulæ:

$$r^n - \left(\frac{s}{a}\right) r + \frac{s}{a} - 1 = 0:$$

$$r^n - \left(\frac{s}{s-l}\right) r^{n-1} + \frac{l}{s-l} = 0:$$

$$(s-a)\,a^{\frac{1}{n-1}} = (s-l)\,l^{\frac{1}{n-1}}:$$

from the two former of which, r may be found in terms of the rest involved with it, by the solutions of equations of n dimensions.

203. *To insert* m *geometric means between* a *and* b.

Retaining the preceding notation, we have

$$n = m + 2, \text{ and } \therefore\ n - 1 = m + 1:$$

$$\text{whence, } r = \left(\frac{b}{a}\right)^{\frac{1}{n-1}} = \left(\frac{b}{a}\right)^{\frac{1}{m+1}}, \text{ is found}:$$

and thus, the means ar, ar^2, &c., ar^m will be immediately determined.

Ex. Insert three geometric means between 1 and 16.

Here, we have $r = (16)^{\frac{1}{4}} = 4^{\frac{1}{2}} = 2$:

$\therefore$ the required means are 2, 4, and 8:

and the completed series is 1, 2, 4, 8 and 16,

which form a regular geometrical progression.

In cases of this kind, all the symbolical values of r are rejected as of no use, though they would also answer the question.

204. Cor. This evidently finds the series, from having given the two extremes and the number of terms: the terms of which will be

$$a,\ (a^{n-2}b)^{\frac{1}{n-1}},\ (a^{n-3}b^2)^{\frac{1}{n-1}},\ \&c.,\ (a^2b^{n-3})^{\frac{1}{n-1}},\ (ab^{n-2})^{\frac{1}{n-1}},\ b:$$

also, the m geometric means between a and b, will be

$$(a^m b)^{\frac{1}{m+1}},\ (a^{m-1}b^2)^{\frac{1}{m+1}},\ \&c.,\ (a^2b^{m-1})^{\frac{1}{m+1}},\ (ab^m)^{\frac{1}{m+1}}.$$

205. *To find an expression for the sum of an infinite geometrical progression.*

When r is a proper fraction, it is evident that each of the terms ar, ar^2, &c., ar^{n-1}, is less than that which immediately precedes it, but that no term can ever become $= 0$, so long as n remains finite: if, however, n be supposed to become indefinitely great, the terms at length become of unassignable magnitude, and the sum of the series will admit of a *limit* beyond which it cannot pass. in other words, what is usually termed the sum of the series continued *in infinitum* will be finite: for, since

$$s = \frac{ar^n - a}{r - 1} = \frac{a - ar^n}{1 - r} = \frac{a}{1 - r} - \frac{ar^n}{1 - r}:$$

and when r is a proper fraction and n indefinitely great, r^n becomes indefinitely small, or small compared to any finite quantity whatever: and the latter term $\frac{ar^n}{1-r}$ may be neglected in comparison with the former $\frac{a}{1-r}$: whence, if σ denote the limit of the sum of the series, we shall have

$$\sigma = \frac{a}{1 - r}.$$

Ex. 1. Find the sum of $1 + \frac{1}{2} + \frac{1}{4} +$ &c. *in infinitum.*

Here, $a = 1$, and $r = \frac{1}{2}$: $\therefore\ \sigma = \frac{1}{1 - \frac{1}{2}} = 2.$

In this instance, 2 is the number to which the sum of the series *continually* approaches, by the increase of the number of its terms; *towards* which it may come nearer than by any *assignable* difference; and *beyond* which it can never pass: thus, by actual addition, we shall find,

$$\text{the sum of two terms} = 1 + \frac{1}{2} = \frac{3}{2} = 2 - \frac{1}{2}:$$

$$\text{three} \quad = \frac{3}{2} + \frac{1}{4} = \frac{7}{4} = 2 - \frac{1}{4}:$$

$$\text{four} \quad = \frac{7}{4} + \frac{1}{8} = \frac{15}{8} = 2 - \frac{1}{8}:$$

$$\text{five} \quad = \frac{15}{8} + \frac{1}{16} = \frac{31}{16} = 2 - \frac{1}{16}.$$

$$\text{six} \quad = \frac{31}{16} + \frac{1}{32} = \frac{63}{32} = 2 - \frac{1}{32}:$$

$$\text{\&c.} \quad = \text{\&c.} = \text{\&c.} = \text{\&c.}:$$

and it will be observed: that the sum becomes at each step more and more nearly equal to 2, as determined by the formula; that no finite number of terms can ever amount to 2; and that the sum of the series at length differs from 2, by less than any assignable quantity: whence it follows that, in all computations conducted under such circumstances, 2 may be regarded as *equivalent* to the sum of $1 + \frac{1}{2} + \frac{1}{4} +$ &c. *in infinitum.*

Ex. 2. Required the limit of the sum of the series,

$$\frac{1}{3} - \frac{1}{3.2} + \frac{1}{3.2^2} - \frac{1}{3.2^3} + \text{\&c.}$$

Here, $a = \frac{1}{3}$ and $r = -\frac{1}{2}$: $\therefore \sigma = \frac{2}{9}$:

and the conditions of the last example will be found to hold good in this.

206. Cor. Whenever it may be necessary to use them, we shall have the following equalities:

$$a = (1 - r)\,\sigma, \text{ and } r = \frac{\sigma - a}{\sigma}.$$

207. *In an infinite geometrical series, find under what circumstances any term is greater than, equal to, or less than the sum of all the terms that succeed it.*

Let the series be a, ar, ar^2, &c., ar^{n-1}, ar^n, &c.: then,

$$\text{if } ar^{n-1} > = < ar^n + ar^{n+1} + ar^{n+2} + \text{\&c. in infinitum}$$

$$> = < ar^n (1 + r + r^2 + \text{\&c. in infinitum})$$

$$> = < ar^n \left(\frac{1}{1-r}\right)$$

$$> = < \frac{ar^n}{1-r}:$$

$$\text{we shall have } 1 > = < \frac{r}{1-r}.$$

(1) If any term of the series be greater than the sum of all that follow it, 1 is greater than $\frac{r}{1-r}$:

$$\therefore\ 1 - r > r, \quad 1 > 2r, \text{ and } \tfrac{1}{2} > r:$$

that is, r is less than $\frac{1}{2}$: and conversely.

(2) If any term of the series be equal to the sum of all the terms that follow it, we have $1 = \frac{r}{1-r}$: and $\therefore\ r = \frac{1}{2}$: and conversely.

(3) If any term of the series be less than the sum of all that follow it, 1 is less than $\frac{r}{1-r}$:

$$\therefore\ 1 - r < r, \quad 1 < 2r, \text{ and } \tfrac{1}{2} < r:$$

that is, r is greater than $\frac{1}{2}$: and conversely.

208. Cor. 1. Hence also in any other series not geometrical, the same conclusions will hold good, where r is not less than the inverse ratio of any two consecutive terms.

209. Cor. 2. Whenever the inverse ratio of any two consecutive coefficients of x, in the series

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \&c.$$

is a finite quantity, it will always be possible to assume x so small, that any one term of the series may exceed the sum of all those which follow it.

For, if ρ be the greatest of these ratios, then

$$\begin{aligned} a_1 x + a_2 x^2 + \&c. &= a_1 \left(x + \frac{a_2}{a_1} x^2 + \frac{a_3}{a_1} x^3 + \&c.\right) \\ &< a_1 x \,(1 + \rho x + \rho x^2 + \&c.) : \end{aligned}$$

whence, if we have

$$\begin{aligned} a_1 x + a_2 x^2 + \&c. &< a_1 x \,(1 + \rho x + \rho^2 x^2 + \&c.) \\ &< a_1 x \left(\frac{1}{1 - \rho x}\right) : \end{aligned}$$

it is evident that the first term will be greater than the sum of all that follow it,

$$\text{when } a_0 = \text{or} > \frac{a_1 x}{1 - \rho x} :$$

$$\text{when } a_0 - a_0 \rho x = \text{or} > a_1 x :$$

$$\text{when } a_0 = \text{or} > (a_0 \rho + a_1)\, x :$$

$$\text{when } x = \text{or} < \frac{a_0}{a_0 \rho + a_1} :$$

$$\text{when } x = \text{or} < \frac{1}{\rho + \dfrac{a_1}{a_0}} = \text{or} < \frac{1}{2\rho}\,.$$

A similar proof will be applicable, whatever term may be assumed as the first.

210. *To find the sum of a series of quantities in geometrical progression, having their coefficients in arithmetical progression.*

Assume,

$$a + (a + b) r + \&c. + \{a + (n - 2) b\} r^{n-2}$$
$$+ \{a + (n - 1) b\} r^{n-1} = s:$$
$$\therefore\ ar + (a + b) r^2 + \&c. + \{a + (n - 2) b\} r^{n-1}$$
$$+ \{a + (n - 1) b\} r^n = rs:$$

whence, by subtraction, we obtain

$$a + br + br^2 + \&c. + br^{n-1} - \{a + (n - 1) b\} r^n = - (r - 1) s:$$

$$\therefore\ a + \frac{br (r^{n-1} - 1)}{r - 1} - \{a + (n - 1) b\} r^n = - (r - 1) s:$$

$$\text{and } s = \frac{\{a + (n - 1) b\} r^n - a - \dfrac{br (r^{n-1} - 1)}{r - 1}}{r - 1}$$

$$= \frac{\{a + (n - 1) b\} r^n - a}{r - 1} - \frac{br (r^{n-1} - 1)}{(r - 1)^2}.$$

When r is a fraction, the sum of a series of fractions, whose numerators are in arithmetical, and denominators in geometrical progression, may be obtained by this formula.

211. Cor. If r be a proper fraction, and the number of terms be supposed indefinitely great, the sum will admit of a limit expressed by

$$\sigma = \frac{a}{1 - r} + \frac{br}{(1 - r)^2} = \frac{a - (a - b) r}{(1 - r)^2},$$

the terms involving r^n being neglected, as in article (205).

Ex. 1. Required the sum of the series $1 + 2x + 3x^2 + \&c.$ to n terms, and *in infinitum* when possible.

Here, $1 + 2x + 3x^2 + \&c. + (n - 1) x^{n-2} + nx^{n-1} = s:$

$$\therefore\ x + 2x^2 + 3x^3 + \&c. + (n - 1) x^{n-1} + nx^n = xs:$$

whence, by subtraction, we obtain

$$1 + x + x^2 + \&c. + x^{n-1} - nx^n = s(1 - x):$$

$$\therefore\ s = \frac{nx^n}{x-1} - \frac{x^n - 1}{(x-1)^2} = \frac{nx^{n+1} - (n+1)x^n + 1}{(x-1)^2}.$$

When x is a proper fraction, and n indefinitely great, x^{n+1} and x^n become indefinitely small, so that $\sigma = \dfrac{1}{(1-x)^2}$.

Ex. 2. Find the sum of the series $1\,\frac{1}{2} + 2\,.\,\frac{1}{2^2} + 3\,.\,\frac{1}{2^3} + \&c.$ to n terms.

Here, $1\,.\,\frac{1}{2} + 2\,.\,\frac{1}{2^2} + \&c. + (n-1)\,.\,\frac{1}{2^{n-1}} + n\,.\,\frac{1}{2^n} = s:$

$$\therefore\ 1\,.\,\frac{1}{2^2} + 2\,.\,\frac{1}{2^3} + \&c. + (n-1)\,.\,\frac{1}{2^n} + n\,.\,\frac{1}{2^{n+1}} = \frac{1}{2}s:$$

whence, by subtraction, we find immediately

$$\frac{1}{2}s = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \&c. + \frac{1}{2^n} - \frac{n}{2^{n+1}}$$

$$= 1 - \frac{1}{2^n} - \frac{n}{2^{n+1}} = \frac{2^{n+1} - (n+2)}{2^{n+1}}:$$

$$\text{and therefore } s = \frac{2^{n+1} - (n+2)}{2^n}.$$

If n be infinite, we shall have $\sigma = 2$.

HARMONICAL PROGRESSION.

212. Def. An Harmonical Progression is a series of quantities in continued harmonical proportion: or such that if any three consecutive terms be taken, the first has to the third the same ratio, which the difference of the first and second has to the difference of the second and third, as appears from article (184).

Thus, if a, b, c, d, &c. be the consecutive terms of an harmonical progression, we shall have

$$a : c = a - b : b - c:$$

$$b : d = b - c : c - d: \text{ \&c.}$$

213. *The reciprocals of the terms of an harmonical progression, are in arithmetical progression.*

Let a, b, c, d, e, &c. be the terms of the series:

$$\text{then, } a : c = a - b : b - c:$$

$$\therefore ab - ac = ac - bc, \text{ and } \frac{ab}{abc} - \frac{ac}{abc} = \frac{ac}{abc} - \frac{bc}{abc}:$$

$$\text{that is, } \frac{1}{c} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a}, \text{ and } \therefore \frac{1}{a} - \frac{1}{b} = \frac{1}{b} - \frac{1}{c}:$$

$$\text{similarly, } \frac{1}{b} - \frac{1}{c} = \frac{1}{c} - \frac{1}{d}: \quad \frac{1}{c} - \frac{1}{d} = \frac{1}{d} - \frac{1}{e}: \text{ \&c.}$$

$$\text{whence, } \frac{1}{a} - \frac{1}{b} = \frac{1}{b} - \frac{1}{c} = \frac{1}{c} - \frac{1}{d} = \frac{1}{d} - \frac{1}{e} = \text{\&c.}$$

$$\text{or } \frac{1}{a}, \ \frac{1}{b}, \ \frac{1}{c}, \ \frac{1}{d}, \ \frac{1}{e}, \text{ \&c.}$$

are equidifferent, and therefore in arithmetical progression.

The converse is easily proved: and it hence appears that the terms of an harmonical progression, taken at equal intervals, are also in harmonical progression.

214. Cor. If x, y be any two adjacent terms of the harmonical series a, b, c, &c., x, y, &c.:

$$\text{then, } \frac{1}{b} - \frac{1}{a} = \frac{1}{y} - \frac{1}{x}, \text{ or } \frac{a-b}{ab} = \frac{x-y}{xy}:$$

$$\text{that is, } ab : xy = a - b : x - y:$$

or, the product of the first two terms is to the product of any two adjacent terms, as the difference between the first two is to the difference between the other two.

215. *Given the first two terms of an harmonical progression, to find the* n$^{\text{th}}$ *term.*

Let a and b be the first two terms, l the n^{th} term; then $\frac{1}{l}$ is the n^{th} term of an arithmetical progression, whose first two terms are $\frac{1}{a}$ and $\frac{1}{b}$:

$$\therefore \text{ the common difference} = \frac{1}{b} - \frac{1}{a} = \frac{a-b}{ab}:$$

$$\text{whence, } \frac{1}{l} = \frac{1}{a} + (n-1)\frac{a-b}{ab} = \frac{(n-1)a-(n-2)b}{ab}:$$

$$\text{and therefore } l = \frac{ab}{(n-1)a-(n-2)b}.$$

Making n equal to 1, 2, 3, 4, &c. in succession, we shall be able to continue the series as far as we please: also, if b be considered the first term and a the second, the series may be continued backwards by the same formula.

Ex. If the first two terms be 1 and $\frac{1}{2}$, the series continued forward will be found to be

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \text{ \&c.}:$$

and continued the other way, it will be

$$\frac{1}{2}, 1, \infty, -1, -\frac{1}{2}, -\frac{1}{3}, \text{ \&c.}$$

The sum of an harmonical series cannot be expressed by means of a simple algebraical formula, like those found in arithmetical and geometrical progression.

216. *To insert* m *harmonic means between* a *and* b.

Here, if d be the common difference of the reciprocals of the terms, we have

$$\frac{1}{b} = \frac{1}{a} + (n-1)d, \text{ and } \therefore d = \frac{a-b}{(n-1)ab} = \frac{a-b}{(m+1)ab}:$$

whence, the arithmetical progression is found: and by inverting its terms, the harmonical means will be ascertained.

This process determines the harmonical series, when the two extremes and the number of terms are given.

Ex. Insert two harmonic means between 3 and 12.

Here, $\frac{1}{12} = \frac{1}{3} + 3d$, and $\therefore d = -\frac{1}{12}$:

$\therefore$ the arithmetic means are $\frac{1}{3} - \frac{1}{12} = \frac{1}{4}$, and $\frac{1}{4} - \frac{1}{12} = \frac{1}{6}$:

whence, the harmonic means are 4 and 6: and 3, 4, 6, 12 form an harmonical progression.

217. The formulæ investigated in this chapter will be now applied and illustrated in the following miscellaneous questions.

(1) If 100 stones be placed in a straight line, exactly a yard asunder, the first being one yard from a basket: what distance will a person go, who gathers them up singly, returning with each to the basket?

Since he goes 2 yards for the first stone, 4 for the second, 6 for the third, &c., the distances travelled form an arithmetical progression whose first term = 2, common difference = 2, and number of terms = 100: whence, we have

$$\text{the required space} = (4 + 198)\,50 = 202 \times 50$$

$$= 10100 \text{ yards} = 5 \text{ miles } 1300 \text{ yards}$$

(2) If a number of workmen be two days in raising the tenth foot of a tower which is to be a hundred feet high: how long will they be in building the tower, the time of raising any foot being in proportion to its height?

Here, $10 : p = 2 : \frac{p}{5}$ = the time of raising the p^{th} foot:

whence, assigning to p, the values 1, 2, 3, &c. 100, we have the

whole time $= \frac{1}{5}(1 + 2 + 3 + \text{\&c. to 100 terms})$

$$= \frac{1}{5}(101 \times 50) = 1010 \text{ days.}$$

(3) To find the sum of n terms of the series

$$a^2, \quad (a+d)^2, \quad (a+2d)^2, \text{ \&c.}$$

Taking α, β, γ, δ, &c. κ, λ to represent the terms of the series a, $a+d$, $a+2d$, &c., we have

$$\beta^3 - \alpha^3 = (a+d)^3 - a^3 = 3a^2 d + 3ad^2 + d^3:$$

$$\gamma^3 - \beta^3 = (\beta+d)^3 - \beta^3 = 3\beta^2 d + 3\beta d^2 + d^3:$$

$$\delta^3 - \gamma^3 = (\gamma+d)^3 - \gamma^3 = 3\gamma^2 d + 3\gamma d^2 + d^3:$$

$$\text{\&c.} = \text{\&c.} = \text{\&c.}$$

$$\lambda^3 - \kappa^3 = (\kappa+d)^3 - \kappa^3 = 3\kappa^2 d + 3\kappa d^2 + d^3:$$

$$(\lambda+d)^3 - \lambda^3 = 3\lambda^2 d + 3\lambda d^2 + d^3:$$

whence, adding together the vertical rows, we find

$$(\lambda+d)^3 - a^3$$

$$= 3d(\alpha^2 + \beta^2 + \gamma^2 + \text{\&c.} + \lambda^2) + 3d^2(\alpha + \beta + \gamma + \text{\&c.} + \lambda) + nd^3$$

$$= 3d \text{ (the required sum)} + 3d^2 \{2a + (n-1)d\} \frac{n}{2} + nd^3:$$

$\therefore$ the required sum

$$= \frac{(a+nd)^3 - a^3}{3d} - \{2a + (n-1)d\} \frac{nd}{2} - \frac{nd^2}{3}$$

$$= \{6a^2 + 6(n-1)ad + (n-1)(2n-1)d^2\} \frac{n}{1.2.3}.$$

If $a = 1 = d$, we have the sum of n terms of the series

$$1^2 + 2^2 + 3^2 + \text{\&c.} + n^2$$

$$= \{6 + 6n - 6 + (n-1)(2n-1)\} \frac{n}{1.2.3} = \frac{n(n+1)(2n+1)}{1.2.3}.$$

This principle might be extended to higher powers, but easier methods will be given in the first Appendix.

(4) If $\dfrac{a+bx}{a-bx} = \dfrac{b+cx}{b-cx} = \dfrac{c+dx}{c-dx} = \text{\&c.}$: then will a, b, c, d, &c., be in geometrical progression.

From, $\dfrac{a+bx}{a-bx}=\dfrac{b+cx}{b-cx}$, we find $b=\sqrt{ac}$:

from, $\dfrac{b+cx}{b-cx}=\dfrac{c+dx}{c-dx}$, we find $c=\sqrt{bd}$: &c.

therefore each of them being a geometric mean between the two adjacent to it, the quantities will form a geometrical progression.

(5) If p be the product, s the sum, and s' the sum of the reciprocals of n quantities, in geometrical progression: then will $p^2 = \left(\dfrac{s}{s'}\right)^n$.

Let x, xy, xy^2, &c., xy^{n-1} represent the quantities: then,

$$p = x \times xy \times xy^2 \times \text{\&c.} \times xy^{n-1} = x^n y^{\frac{1}{2}n(n-1)}:$$

$$s = \frac{x(y^n-1)}{y-1}, \quad s' = \frac{\dfrac{1}{x}\left\{\left(\dfrac{1}{y}\right)^n - 1\right\}}{\dfrac{1}{y}-1} = \frac{1}{xy^{n-1}}\left(\frac{y^n-1}{y-1}\right):$$

$$\text{whence, } \left(\frac{s}{s'}\right)^n = (x^2y^{n-1})^n = x^{2n}y^{n(n-1)} = p^2.$$

(6) If r, r' be the ratios of two geometrical progressions whose first terms are equal: the difference of the sums of n terms will be divisible by $r-r'$.

Here, $s-s' = a\left\{\dfrac{r^n-1}{r-1}-\dfrac{r'^n-1}{r'-1}\right\}$

$$= \frac{a}{(r-1)(r'-1)}\{rr'(r^{n-1}-r'^{n-1})-(r^n-r'^n)+(r-r')\},$$

which by article (35), is divisible by $r-r'$.

This will immediately appear by taking the differences of the corresponding terms of the two series, inasmuch as each of these differences will be of the form $r^m - r'^m$, which is always divisible by $r-r'$, when m is a whole number.

(7) The sums of two infinite geometrical progressions beginning from 1, are σ_1 and σ_2: it is required to prove that the sum of the series, formed by multiplying together their corresponding terms, is $\dfrac{\sigma_1\sigma_2}{\sigma_1+\sigma_2-1}$.

If r_1 and r_2 be the common ratios, we have

$$\sigma_1 = 1 + r_1 + r_1^2 + \&c. \textit{ in infinitum } = \frac{1}{1-r_1}.$$

$$\sigma_2 = 1 + r_2 + r_2^2 + \&c. \textit{ in infinitum } = \frac{1}{1-r_2}:$$

and the sum of the series, resulting from the multiplication of the corresponding terms of these two,

$$= 1 + r_1r_2 + (r_1r_2)^2 + \&c. \textit{ in infinitum}$$

$$= \frac{1}{1-r_1r_2} = \frac{\sigma_1\sigma_2}{\sigma_1+\sigma_2-1}, \text{ by substitution.}$$

(8) Find the value of the recurring decimal .2525 &c.

$$\text{The decimal } = \frac{25}{10^2} + \frac{25}{10^4} + \frac{25}{10^6} + \&c. \textit{ in infinitum}:$$

$$\text{whence, } \sigma = \frac{a}{1-r} = \frac{25}{10^2} \div \left(1 - \frac{1}{10^2}\right) = 25 \div (100-1) = \frac{25}{99}.$$

Similarly, if ab represent the *period* of the decimal:

$$.ab\,ab\,ab \;\&c. = \frac{a}{10} + \frac{b}{10^2} + \frac{a}{10^3} + \frac{b}{10^4} + \&c. \textit{ in infinitum}$$

$$= \frac{10a+b}{99}: \text{ and so of others.}$$

CHAPTER IX.

VARIATIONS, PERMUTATIONS, AND COMBINATIONS.

VARIATIONS AND PERMUTATIONS.

218. Def. The Variations and Permutations of any number of things, are the different orders which can be formed out of them with regard to position, when a certain number and the whole are respectively taken at a time.

Thus, of the three things represented by a, b, c, the variations formed by taking *one* at a time are

$$a, \quad b, \quad c:$$

and when taken *two* and *two* together, the variations are

$$ab, \quad ba, \quad ac, \quad ca, \quad bc, \quad cb:$$

whereas, the permutations formed by taking them *all* together, will be

$$abc, \quad acb, \quad bac, \quad bca, \quad cab, \quad cba.$$

Without attending to the distinction above noticed, the words *Variations*, *Permutations*, *Alternations*, and *Changes*, are often used promiscuously, whether the whole or part be taken at a time: but we shall at present adhere to the definitions just laid down.

219. Cor. If we have *four* things a, b, c, d: their variations, taken *one* at a time, are a, b, c, d: also, of the three things b, c, d, the variations, taken one at a time, are b, c, d: wherefore, if a be placed before each of these, we shall have the variations

$$ab, \quad ac, \quad ad,$$

two being taken at a time, in which a stands first: similarly, we have the variations

$$ba, \quad bc, \quad bd,$$

taken *two* together, in which b stands first:

$$ca, \quad cb, \quad cd,$$

taken *two* together, in which c stands first:

$$da, \quad db, \quad dc,$$

taken *two* together, in which d stands first: and thus the whole of the variations of four things, taken *two* together, is obtained: the number of which is evidently 4×3 or 12.

By similar reasoning, the variations, when *three* are taken together, may be exhibited, and their number found: and so on, as in the following article.

220. *To find the number of variations of* m *things, when* r *of them are always taken together.*

Let the m things be represented by a, b, c, d, &c.: then, the number of their variations, when taken *singly*, will manifestly be m:

and, if we leave out a, there will be $m - 1$ things remaining, the number of whose variations, taken *singly*, will be $m - 1$:

therefore, if a be placed before each of these $m - 1$ things, the number of variations, taken *two* together, in which a stands first, will be $m - 1$:

similarly, the number of variations, taken *two* together, in which each of the things b, c, d, &c. stands first, will be $m - 1$:

therefore, upon the whole, there will be $m(m-1)$ variations of m things, taken *two* together.

Again, if a be left out, there will be $m - 1$ things remaining, the number of whose variations, taken *two* together, is $(m-1)(m-2)$, by what precedes:

whence, if a be placed before each of these variations, there will evidently be $(m-1)(m-2)$ variations, taken *three* together, in which a stands first: and the same may be said of b, c, d, &c.: therefore, upon the whole, there will be $m(m-1)(m-2)$ variations of m things, taken *three* together.

Similarly, it may be shewn from what is done above, that the number of variations of m things, taken *four* together, is

$$m(m-1)(m-2)(m-3).$$

In order to prove that the law observed in the preceding cases is general, let us suppose that the number of variations of m things taken $r-1$ together is

$$m(m-1)(m-2) \text{ &c. } (m-r+2):$$

then, leaving out a, we have $m-1$ things remaining, so that substituting $m-1$ for m in this formula, the number of variations of $m-1$ things taken $r-1$ together, will be

$$(m-1)(m-2)(m-3) \text{ &c. } (m-1-r+2)$$
$$= (m-1)(m-2)(m-3) \text{ &c. } (m-r+1):$$

and placing a before each of these variations of $(m-1)$ things, taken $(r-1)$ together, we shall evidently have

$$(m-1)(m-2)(m-3) \text{ &c. } (m-r+1)$$

variations of m things, taken r together, in which a stands first: and the same being true when b, c, d, &c. stand first, the number of variations of m things taken r together will be expressed by

$$m(m-1)(m-2) \text{ &c. } (m-r+1),$$

which is the formula above assumed, with r in the place of $r-1$.

Whence we infer, that if the assumed formula hold good for any one value of r, it will be true for the next superior value: and it having been demonstrated to be true, when the values of r are 1, 2, 3, it must be true when $r=4$; therefore when $r=5$; therefore when $r=6$, &c.; and by successive inductions, when r is any number whatever not greater than m.

Denoting the number of variations of m things taken r together by V_r, we shall have

$$V_r = m(m-1)(m-2) \text{ \&c. } (m-r+1),$$

which is sometimes termed the number of variations of m things *without repetitions*, of the r^{th} class.

Ex. If there be six things, we shall have

$$V_1 = 6,\ V_2 = 6.5 = 30,\ V_3 = 6.5.4 = 120,$$

$$V_4 = 6.5.4.3 = 360,\ V_6 = 6.5.4.3.2 = 720.$$

221. When $r = m$, or all the things are taken together each time, we shall have

$$V_m = m(m-1)(m-2) \text{ \&c. } 3.2.1$$

$$= 1.2.3. \text{ \&c.}.m:$$

which is the number of *Permutations* of m things, agreeably to the Definition at the head of the Chapter: and it is, in fact, the *greatest* number of variations the things admit of.

Ex. Find the number of changes which may be rung upon seven bells, taken all together.

The number of changes required is evidently the same as the number of permutations formed out of seven things, and is therefore

$$= 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 = 5040.$$

COMBINATIONS.

222. Def. The Combinations of any number of things, are the different collections that can be formed out of them, by taking a certain number at a time, without regard to the order in which they are arranged.

Thus, of a, b, c there will be *three things*, a, b, c, formed by taking *one* at a time: *three combinations*, ab, ac, bc, formed by taking *two* at a time, and *one combination* abc made by taking all the *three* together.

223. *To find the number of combinations that can be formed out of* m *things, by always taking* r *of them together.*

Let C_r denote the number of combinations that can be formed out of m things taken r together: V_r the corresponding number of variations:

then, since every combination of r things taken all together, admits of $1.2.3.$ &c. r permutations, by article (221), we shall manifestly have $1.2.3.$ &c. $r \times$ the number of combinations equal to the corresponding number of variations: that is,

$$(1.2.3. \text{ \&c. } r)\, C_r = m(m-1)(m-2) \text{ \&c. } (m-r+1):$$

$$\text{whence,} \quad C_r = \frac{m(m-1)(m-2) \text{ \&c. } (m-r+1)}{1.2.3. \text{ \&c. } r}.$$

This is sometimes called the number of combinations *without repetitions*, of the r^{th} class.

224. Cor. The reasoning of the last article may be made more clear by means of particular cases.

For, if *two* things be always taken together, the number of variations of m things has been shewn to be $m(m-1)$:

but every combination as ab, admits of *two* variations ab and ba: and therefore there are *twice* as many variations as combinations of this class.

Similarly, out of one combination of *three* things, as abc, it is possible to form $3.2.1$ variations, by article (221): and therefore the number of variations of any number of things, taken *three* together, will manifestly be $1.2.3$ or *six* times as great as the corresponding number of combinations: and so on.

Ex. If we make r equal to the numbers 1, 2, 3, &c., m, in order, we shall have

$$C_1 = m:$$

$$C_2 = \frac{m(m-1)}{1.2}:$$

$$C_3 = \frac{m(m-1)(m-2)}{1.2.3}: \text{ \&c.}$$

$$C_m = \frac{m(m-1)(m-2) \text{ \&c. } 3.2.1}{1.2.3. \text{ \&c. } m} = 1.$$

225. If in the general formula established in article (223), we substitute $m-r$ for r, we shall have

$$C_{m-r} = \frac{m(m-1)(m-2)\,\&\text{c.}\,(r+1)}{1.2.3.\,\&\text{c.}\,(m-r)}$$

$$= \frac{m(m-1)(m-2)\,\&\text{c.}\,(r+1)\,r\,.\,\&\text{c.}\,.\,3.2.1}{1.2.3.\,\&\text{c.}\,r\,.\,1.2.3.\,\&\text{c.}\,(m-r)}$$

$$= \frac{m(m-1)(m-2)\,\&\text{c.}\,(m-r+1)}{1.2.3.\,\&\text{c.}\,r} \times \frac{(m-r)\,\&\text{c.}\,3.2.1}{1.2.3.\,\&\text{c.}\,(m-r)}$$

$$= \frac{m(m-1)(m-2)\,\&\text{c.}\,(m-r+1)}{1.2.3.\,\&\text{c.}\,r}$$

$$= C_r.$$

The combinations belonging to the respective sets denoted by C_r and C_{m-r}, are said to be *supplementary* to each other.

Ex. If $m=7$, it will be seen immediately that

$$C_1 = 7 = C_6:\ C_2 = 21 = C_5:\ C_3 = 35 = C_4.$$

226. *To find how many things must be taken together, that the number of combinations may be the greatest possible.*

Let C_r denote the number of combinations required:

then, it is evident from the nature of the case, that C_r must not be less than either C_{r-1} or C_{r+1}: and consequently that

$$\frac{C_r}{C_{r-1}} \text{ and } \frac{C_r}{C_{r+1}}$$

must neither of them be less than 1:

$$\text{but } \frac{C_r}{C_{r-1}} = \frac{m-r+1}{r}, \text{ and } \frac{C_r}{C_{r+1}} = \frac{r+1}{m-r}:$$

$$\therefore \frac{m-r+1}{r} \text{ is not } < 1, \text{ and } \frac{r+1}{m-r} \text{ is not } < 1:$$

$$\therefore m-r+1 \text{ is not } < r, \text{ and } r+1 \text{ is not } < m-r:$$

$$\therefore m+1 \text{ is not } < 2r, \text{ and } 2r \text{ is not } < m-1:$$

$$\text{that is, } 2r \text{ is not } > m+1, \text{ nor } < m-1:$$

$$\text{and } \therefore\ r \text{ is not } > \tfrac{1}{2}(m+1), \text{ nor } < \tfrac{1}{2}(m-1).$$

If m be odd and $= 2p+1$, r is not $> p+1$ nor $< p$;

$$\text{and } r \text{ may therefore } = p+1, \text{ or } p,$$

$$\text{or } = \tfrac{1}{2}(m+1), \text{ or } \tfrac{1}{2}(m-1).$$

If m be even and $= 2p$, r is not $> p + \frac{1}{2}$ nor $< p - \frac{1}{2}$;

$$\text{and } r \text{ must therefore } = p = \tfrac{1}{2}m.$$

227. Cor. From this investigation it appears that, when the number of things is odd, the greatest number of combinations may be obtained in two ways, which give the same result: but that when it is even, there is only one set of combinations which will answer the purpose.

When m is odd, we shall have

$$C_r = \frac{m(m-1)(m-2)\ \&\text{c.}\ \frac{1}{2}(m+1)}{1\,.\,2\,.\,3\,.\ \&\text{c.}\ \frac{1}{2}(m+1)}:$$

$$\text{or, } = \frac{m(m-1)(m-2)\ \&\text{c.}\ \frac{1}{2}(m+3)}{1\,.\,2\,.\,3\,.\ \&\text{c.}\ \frac{1}{2}(m-1)}:$$

and when m is even, we shall have

$$C_r = \frac{m(m-1)(m-2)\ \&\text{c.}\ (\frac{1}{2}m+1)}{1\,.\,2\,.\,3\,.\ \&\text{c.}\ \frac{1}{2}m}.$$

Ex. Of five things, find how many must be taken together, that the number of combinations may be the greatest possible.

Here, $m = 5$, and therefore the required number

$$= \tfrac{1}{2}(m+1) = 3, \text{ or } = \tfrac{1}{2}(m-1) = 2:$$

and the corresponding number of combinations $= 10$.

228. *To find the number of different permutations, which can be formed out of* m *things taken all together, when* p *are of one sort,* q *of another; &c.*

Let P denote the number of permutations required: and first suppose that of the m things, p only are identical: then,

if the p things were all different, they would admit of $1 . 2 . 3 .$ &c. p permutations instead of 1, as they actually do: whence, $(1 . 2 . 3 . \&c. p) P =$ the number of permutations formed out of m things all different

$$= 1 . 2 . 3 . \&c. m:$$

$$\text{therefore, } P = \frac{1 . 2 . 3 . \&c. m}{1 . 2 . 3 . \&c. p}.$$

Again, if in addition to this, q things are also identical, and P denote the required number of permutations, we shall have by the same mode of reasoning,

$$(1 . 2 . 3 . \&c. p)(1 . 2 . 3 . \&c. q) P$$

= the number of permutations of m things all different

$$= 1 . 2 . 3 . \&c. m:$$

$$\text{whence, } P = \frac{1 . 2 . 3 . \&c. m}{(1 . 2 . 3 . \&c. p)(1 . 2 . 3 . \&c. q)}:$$

and a similar formula will manifestly be correct, whatever be the number of quantities p, q, &c.

Ex. 1. Required the number of different permutations, that can be formed out of the letters of the word *Difference*.

Here, $m = 10$: also, there are 2 f's, or $p = 2$, and 3 e's, or $q = 3$:

$$\text{whence, } P = \frac{1 . 2 . 3 . 4 . 5 . 6 . 7 . 8 . 9 . 10}{(1 . 2)(1 . 2 . 3)} = 302400.$$

Ex. 2. Find the number of different permutations that can be formed out of $a^{m-r} b^r$, when written at length.

Here are m quantities, and a and b being repeated $m - r$ and r times respectively, we have

$$P = \frac{1 . 2 . 3 . \&c. m}{\{1 . 2 . 3 . \&c. (m - r)\} \{1 . 2 . 3 . \&c. r\}}$$

$$= \frac{m(m-1) \&c. (m - r + 1)}{1 . 2 . 3 . \&c. r}:$$

by rejecting from the numerator and denominator, the factors common to both.

229. Cor. From the formula,

$$P = \frac{1 \, . \, 2 \, . \, 3 \, . \, \&c. \, m}{(1 \, . \, 2 \, . \, 3 \, . \, \&c. \, p)\,(1 \, . \, 2 \, . \, 3 \, . \, \&c. \, q)\ \&c.},$$

it is manifest that if p and q change places, the number of permutations will remain the same: as for instance, in the last example, the result will be the same, whether p represent the number of a's, and q the number of b's: or *vice versâ*.

230. *To determine the form of the continued product of the* m *simple factors*, x + a, x + b, x + c, &c.

Here, $(x + a)(x + b) = x^2 + (a + b)x + ab$:

$(x + a)(x + b)(x + c) = x^3 + (a + b + c)x^2 + (ab + ac + bc)x + abc$:

$$(x + a)(x + b)(x + c)(x + d) = x^4 + (a + b + c + d)x^3$$
$$+ (ab + ac + ad + bc + bd + cd)x^2 + (abc + abd + acd + bcd)x + abcd:$$

whence, if this kind of form be assumed to be true for $m - 1$ factors, and the remaining factor be introduced into both its members, it will immediately appear to hold good for m factors.

Wherefore, in the required product, the *first* term will be x^m: the *second* term will be x^{m-1}, having for its coefficient, the sum of the m quantities a, b, c, &c.: the *third* term will be x^{m-2}, having its coefficient equal to the sum of the combinations of the m quantities a, b, c, &c. taken *two* together, the number of which will be $\dfrac{m(m-1)}{1 \, . \, 2}$: and so on: and the r^{th} term will be x^{m-r+1}, with a coefficient equal to the sum of the combinations of the m quantities a, b, c, &c., formed by taking $r-1$ at a time, the number of which will therefore be

$$\frac{m(m-1)(m-2)\ \&c.\ (m-r+2)}{1 \, . \, 2 \, . \, 3 \, . \, \&c. \, (r-1)}:$$

and the last term will be the continued product of the m quantities a, b, c, &c.

231. Cor. If the m quantities a, b, c, &c. be all equal to one another, we shall have

$$(x+a)^m = x^m + max^{m-1} + \frac{m(m-1)}{1.2}a^2x^{m-2} + \text{\&c.},$$

which is the *Binomial Theorem*, in the case where the index is a positive whole number.

232. *To ascertain the nature and form of the continued product,*

$$(x_1 + a_1)(x_2 + a_2)(x_3 + a_3) \text{ \&c. } \textit{to } m \textit{ factors.}$$

Here, we observe, that every term of the result must necessarily consist of m factors, and therefore all the terms will be homogeneous:

also, when $m-1$ of the x's are found in any term, *one* of the a's must be involved with them: when $m-2$ of the x's are involved, *two* of the a's will also be found there: and so on: in other words, the x's and a's are *complementary* to each other, the number of both together being in every term equal to m:

now, from the formula,

$$P = \frac{1.2.3.\text{\&c. } m}{(1.2.3.\text{\&c. } p)(1.2.3.\text{\&c. } q)\text{ \&c.}},$$

the number of terms which are made up of $(m-1)$ x's and *one* a, will be

$$\frac{1.2.3.\text{\&c. } m}{\{1.2.3.\text{\&c. }(m-1)\}\{1\}} = m:$$

the number of terms involving $(m-2)$ x's and *two* a's, will be

$$\frac{1.2.3.\text{\&c. } m}{\{1.2.3.\text{\&c. }(m-2)\}\{1.2\}} = \frac{m(m-1)}{1.2}:$$

the number of terms involving $(m-3)$ x's and *three* a's, will be

$$\frac{1.2.3.\text{\&c. } m}{\{1.2.3.\text{\&c. }(m-3)\}\{1.2.3\}} = \frac{m(m-1)(m-2)}{1.2.3}: \text{ \&c.}$$

whence both the form, and the number of terms, of the product are determined.

233. Cor. If it be required to find the nature, and the number of the terms, of the continued product of m multinomial factors of the same kind, we observe as before that all the terms will be *homogeneous*: and the sum of the indices in every term being m, the magnitudes and number of such terms will be determined in the same manner, from the formula,

$$P = \frac{1 \cdot 2 \cdot 3 \cdot \&c. \; m}{(1 \cdot 2 \cdot 3 \cdot \&c. \; p)\,(1 \cdot 2 \cdot 3 \cdot \&c. \; q)\,(1 \cdot 2 \cdot 3 \cdot \&c. \; r)\;\&c.}:$$

the values of p, q, r, &c. being always such as to satisfy the equation of *condition*,

$$p + q + r + \&c. = m.$$

This amounts to what is generally called the *Multinomial*, or *Polynomial Theorem*.

CHAPTER X.

INDETERMINATE COEFFICIENTS, AND THE BINOMIAL THEOREM.

INDETERMINATE COEFFICIENTS.

234. Def. The method of Indeterminate Coefficients is a process by which the *Expansion* or *Developement* of algebraical expressions may be effected, by assuming for them a series of powers of one of the letters involved, combined with coefficients that are subsequently to be assigned in terms of the rest: and it is of the most extensive utility in algebraical *Analysis*, as will be seen in the following pages.

235. If the equation,

$$A + Bx + Cx^2 + Dx^3 + \&c. = a + bx + cx^2 + dx^3 + \&c.$$

wherein both members are continued at pleasure, be true for all values that can possibly be assigned to x: then will the coefficients of the same powers of x in both members be equal to one another:

$$\text{that is, } A = a, \; B = b, \; C = c, \; D = d, \; \&c.$$

For, since independently of any particular value of x,

$$A + Bx + Cx^2 + Dx^3 + \&c. = a + bx + cx^2 + dx^3 + \&c.:$$

if $x = 0$, we have $A = a$: and there remains

$$Bx + Cx^2 + Dx^3 + \&c. = bx + cx^2 + dx^3 + \&c.:$$

$$\text{or, } B + Cx + Dx^2 + \&c. = b + cx + dx^2 + \&c.:$$

if $x = 0$, we have $B = b$: and by a continuation of this mode of reasoning, it may be similarly demonstrated that

$$C = c, \quad D = d, \text{ \&c.}$$

The truth of this proposition is further manifest from the circumstance, that if we transpose all the terms of the second side of the equation, we shall have

$$(A - a) + (B - b)\,x + (C - c)\,x^2 + (D - d)\,x^3 + \text{\&c.} = 0:$$

which, if the coefficients $A - a$, $B - b$, $C - c$, &c. of the different powers of x were finite, could be satisfied *only* by the *roots* of the equation: and thus, the generality essential to the expression would be destroyed.

236. Cor. 1. If the equation $A + Bx = a + bx$, hold good for any *two* different values of x: then will $A = a$ and $B = b$.

For, let α, β be the two values of x, so that

$$A + B\alpha = a + b\alpha, \text{ and } A + B\beta = a + b\beta:$$

$$\therefore \; B(\alpha - \beta) = b(\alpha - \beta): \text{ whence, } B = b \text{ and } A = a.$$

Similarly, if $A + Bx + Cx^2 = a + bx + cx^2$, be true for any *three* different values of x, it may be shewn that $A = a$, $B = b$, $C = c$: and so on, for any number of terms and correspondent values of x: and this will manifestly lead to the conclusion in the last article.

237. Cor. 2. If $A + Bx + Cx^2 + Dx^3 + \text{\&c.} = 0$, the values of a, b, c, d, &c. being each $= 0$, we shall have

$$A = 0, \; B = 0, \; C = 0, \; D = 0, \text{ \&c.}$$

238. Division and Evolution may be effected by means of indeterminate coefficients, as in the following examples.

Ex. 1. To divide $a + x$ by $1 - bx$, let us assume

$$\frac{a + x}{1 - bx} = A + Bx + Cx^2 + Dx^3 + \text{\&c.}:$$

$\therefore$ multiplying both sides by $1 - bx$, we have

$$\begin{aligned} a + x &= A + Bx + Cx^2 + Dx^3 + \&c. \\ &\qquad - Abx - Bbx^2 - Cbx^3 - \&c. \\ &= A + (B - Ab)x + (C - Bb)x^2 + (D - Cb)x^3 + \&c.: \end{aligned}$$

whence, equating the coefficients of the same powers of x in both members, we have $A = a$:

$$\begin{aligned} B - Ab &= 1, \quad \therefore\ B = 1 + Ab = 1 + ab: \\ C - Bb &= 0, \quad \therefore\ C = Bb = (1 + ab)\,b: \\ D - Cb &= 0, \quad \therefore\ D = Cb = (1 + ab)\,b^2: \\ &\&c. \qquad\qquad \&c. \end{aligned}$$

$$\therefore \frac{a + x}{1 - bx} = a + (1 + ab)\,x + (1 + ab)\,bx^2 + (1 + ab)\,b^2x^3 + \&c.:$$

which is the same as would be obtained by actual division: and the law by which the coefficients are connected is manifest.

Ex. 2. To extract the square root of $1 + x^2$.

Assume $\sqrt{1 + x^2} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \&c.$:

$$\begin{aligned} \therefore\ 1 + x^2 = A^2 &+ ABx + ACx^2 + ADx^3 + AEx^4 + \&c. \\ &+ ABx + B^2x^2 + BCx^3 + BDx^4 + \&c. \\ &\qquad + ACx^2 + BCx^3 + C^2x^4 + \&c. \\ &\qquad\qquad + ADx^3 + BDx^4 + \&c. \\ &\qquad\qquad\qquad + AEx^4 + \&c. \\ &\qquad\qquad\qquad\qquad + \&c. \end{aligned}$$

whence, equating coefficients, we have

$$\begin{aligned} A^2 &= 1, \text{ or } A = 1: \\ 2AB &= 0, \text{ or } B = 0: \\ 2AC + B^2 &= 1, \text{ or } C = \frac{1 - B^2}{2A} = \frac{1}{2}: \\ 2AD + 2BC &= 0, \text{ or } D = -\frac{BC}{A} = 0: \end{aligned}$$

$$2AE + 2BD + C^2 = 0, \text{ or } E = -\frac{2BD + C^2}{2A} = -\frac{1}{8}: \text{ \&c.}$$

$$\therefore \sqrt{1 + x^2} = 1 + \frac{1}{2}x - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \text{\&c.}:$$

and it may be observed that, had we known the form of the expansion beforehand, the odd powers might have been omitted in the assumption, without altering the result.

239. *To determine the form of the continued product of the* m *simple factors*, x + a, x + b, x + c, &c., x + k, x + l.

Let $(x + a)(x + b)$ &c. $(x + l)$ to m factors

$$= x^m + A_1 x^{m-1} + A_2 x^{m-2} + \text{\&c.}:$$

and $(x + b)(x + c)$ &c. $(x + l)$ to $m - 1$ factors

$$= x^{m-1} + B_1 x^{m-2} + B_2 x^{m-3} + \text{\&c.}:$$

then, multiplying both members of the latter assumption by $x + a$, we shall have

$$\begin{aligned} x^m &+ A_1 x^{m-1} + A_2 x^{m-2} + \text{\&c.} \\ &= (x + a)(x^{m-1} + B_1 x^{m-2} + B_2 x^{m-3} + \text{\&c.}) \\ &= x^m + B_1 x^{m-1} + B_2 x^{m-2} + \text{\&c.} \\ &\qquad + a x^{m-1} + a B_1 x^{m-2} + \text{\&c.} \\ &= x^m + (B_1 + a) x^{m-1} + (B_2 + aB_1) x^{m-2} + \text{\&c.}: \end{aligned}$$

$$\text{whence, } A_1 = B_1 + a, \quad A_2 = B_2 + aB_1, \text{ \&c.}:$$

that is, by the introduction of the factor $x + a$, the coefficient of the *second* term is increased by a: and the same being true of all the rest, it follows that $A_1 = a + b + c +$ &c.: by the same operation, the coefficient of the *third* term is increased by aB_1, or by the product of a and the preceding value of $A_1 = a(b + c + d + \text{\&c.})$: whence, we have

$$\begin{aligned} A_2 &= a(b + c + d + \text{\&c.}) \\ &\quad + b(c + d + \text{\&c.}) \\ &\quad + c(d + \text{\&c.}): \end{aligned}$$

$$\text{similarly, } A_3 = ab\,(c + d + \&c.) + ac\,(d + \&c.) + \&c. + bc\,(d + \&c.) + \&c.: \text{ and so on:}$$

from which, we conclude that

A_1 is the sum of a, b, c, &c. taken *singly*:

A_2 is the sum of the products of a, b, c, &c. taken *two* together:

A_3 is the sum of the products of a, b, c, &c. taken *three* together: and similarly, of the succeeding coefficients in order

Hence also, the coefficient of the r^{th} term in the required product, will be the sum of the products of a, b, c, &c. taken $r - 1$ together: and the coefficient of x^r will be the sum of the products of a, b, c, &c., taken $m - r$ at a time.

Ex. Find the continued product of $x + 2$, $x + 6$, $x + 10$ and $x + 14$.

Here, $m = 4$, and therefore the first term is x^4 :

the coefficient of $x^3 = 2+6+10+14 = 32$:

the coefficient of $x^2 = 2.6+2.10+2.14+6.10+6.14+10.14 = 344$:

the coefficient of $x^1 = 2.6.10+2.6.14+2.10.14+6.10.14 = 1408$:

the coefficient of $x^0 = 2.6.10.14 = 1680$:

$$\therefore \text{ the product } = x^4 + 32x^3 + 344x^2 + 1408x + 1680.$$

240. The method of indeterminate coefficients is applied to the resolution of fractions, whose denominators are expressed by factors, into others with simple denominators.

Ex. 1. Resolve $\dfrac{2a - x}{a^2 - x^2}$ into two simple fractions.

Here, $a^2 - x^2 = (a + x)(a - x)$: whence, assuming

$$\frac{2a - x}{a^2 - x^2} = \frac{A}{a + x} + \frac{B}{a - x} = \frac{(A + B)\,a - (A - B)\,x}{a^2 - x^2}:$$

we have $2a - x = (A + B)\,a - (A - B)\,x$:

and equating coefficients, we obtain

$$A + B = 2 \text{ and } A - B = 1:$$

$$\text{whence, } A = \tfrac{3}{2}, \text{ and } B = \tfrac{1}{2}; \text{ so that}$$

$$\frac{2a - x}{a^2 - x^2} \text{ is equivalent to } \frac{3}{2(a + x)} + \frac{1}{2(a - x)}.$$

Ex. 2. Resolve $\dfrac{x^2}{(x + 1)(x + 2)(x + 3)}$ into three simple fractions.

Assume $\dfrac{x^2}{(x + 1)(x + 2)(x + 3)} = \dfrac{A}{x + 1} + \dfrac{B}{x + 2} + \dfrac{C}{x + 3}$

$$= \frac{(A + B + C)x^2 + (5A + 4B + 3C)x + 6A + 3B + 2C}{(x + 1)(x + 2)(x + 3)}:$$

$\therefore\ x^2 = (A + B + C)x^2 + (5A + 4B + 3C)x + 6A + 3B + 2C$, whatever values be assigned to x: whence, we have

$$A + B + C = 1,\quad 5A + 4B + 3C = 0,\quad 6A + 3B + 2C = 0:$$

$$\begin{aligned}
&\text{from (1),} & 5A + 5B + 5C &= 5: \\
&\text{from (2),} & 5A + 4B + 3C &= 0: \\
& & \therefore\ B + 2C &= 5: & (\alpha) \\
&\text{from (1),} & 6A + 6B + 6C &= 6: \\
&\text{from (3),} & 6A + 3B + 2C &= 0: \\
& & \therefore\ 3B + 4C &= 6: & (\beta) \\
&\text{from } (\alpha), & 3B + 6C &= 15: \\
&\text{from } (\beta), & 3B + 4C &= 6:
\end{aligned}$$

$$\therefore\ 2C = 9, \text{ and } C = \tfrac{9}{2}:$$

$$\text{from } (\alpha),\quad B = 5 - 2C = 5 - 9 = -4:$$

$$\text{from (1)},\quad A = 1 - B - C = 1 + 4 - \tfrac{9}{2} = \tfrac{1}{2}:$$

wherefore,

$$\frac{x^2}{(x + 1)(x + 2)(x + 3)} = \frac{1}{2(x + 1)} - \frac{4}{x + 2} + \frac{9}{2(x + 3)}.$$

Ex. 3. Decompose $\dfrac{1}{(x-a)(x-b)(x-c) \text{ \&c. } (x-l)}$ into simple fractions, the number of factors in the denominators being m.

Let

$$\frac{1}{(x-a)(x-b)(x-c) \text{ \&c. } (x-l)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} + \text{\&c.} + \frac{L}{x-l}$$

then, multiplying both members by the denominator of the former, we have

$$\begin{aligned} 1 = {} & A(x-b)(x-c) \text{ \&c. to } (m-1) \text{ factors}, \\ & + B(x-a)(x-c) \text{ \&c. } \ldots\ldots\ldots\ldots \\ & + C(x-a)(x-b) \text{ \&c. } \ldots\ldots\ldots\ldots \\ & + \text{\&c.} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + L(x-a)(x-b) \text{ \&c. } \ldots\ldots\ldots\ldots \end{aligned}$$

which is true for all values of x:

$$\therefore \text{ if } x = a, \quad A = \frac{1}{(a-b)(a-c) \text{ \&c. to } (m-1) \text{ factors}}:$$

$$\text{if } x = b, \quad B = \frac{1}{(b-a)(b-c) \text{ \&c. to } (m-1) \text{ factors}}:$$

$$\text{if } x = c, \quad C = \frac{1}{(c-a)(c-b) \text{ \&c. to } (m-1) \text{ factors}}:$$

$$\text{\&c.} \quad = \quad \text{\&c.}$$

$$\text{if } x = l, \quad L = \frac{1}{(l-a)(l-b) \text{ \&c. to } (m-1) \text{ factors}}:$$

and thus the values of A, B, C, &c. L are determined.

241. Cor. By effecting the multiplications indicated in the last example, we shall have

$$1 = (A + B + C + \text{\&c.} + L)\, x^{m-1} + \text{\&c.}:$$

whence, is obtained $A + B + C + \text{\&c.} + L = 0$:

that is, if a, b, c, &c. k, l be any unequal numbers whatever, then will

$$\frac{1}{(a-b)(a-c)\,\&c.\,(a-l)} + \frac{1}{(b-a)(b-c)\,\&c.\,(b-l)} + \&c.$$

$$+ \frac{1}{(l-a)(l-b)\,\&c.\,(l-k)} = 0.$$

242. This principle may be rendered still more general, and may be extended to indeterminate indices, as well as indeterminate coefficients; so that if we have for every value of x,

$$Ax^{\alpha} + Bx^{\beta} + Cx^{\gamma} + \&c. = A'x^{\alpha'} + B'x^{\beta'} + C'x^{\gamma'} + \&c.:$$

a similar process will lead to the conclusion that

$$\alpha = \alpha',\ \beta = \beta',\ \gamma = \gamma',\ \&c.:\ A = A',\ B = B',\ C = C',\ \&c.$$

THE BINOMIAL THEOREM.

243. Def. The Binomial Theorem is a general Algebraical Formula, by means of which, any power or root of a quantity consisting of two terms, may be expressed by a series of simple quantities: and in its most general form is

$$(a + x)^m$$

$$= a^m + ma^{m-1}x + \frac{m(m-1)}{1.2}a^{m-2}x^2 + \frac{m(m-1)(m-2)}{1.2.3}a^{m-3}x^3 + \&c.,$$

where the quantities a, x and m may be either positive or negative, integral or fractional.

We shall divide the proof of this theorem into the two following propositions.

(1) To determine the law of the formation of the indices, and the coefficient of the second term:

(2) To investigate the law of the formation of the succeeding coefficients:

and since $(a+x)^m = \left\{a\left(1+\frac{x}{a}\right)\right\}^m = a^m\left(1+\frac{x}{a}\right)^m$,

the binomial shall be represented by $1+v$, and its index by m.

244. *To determine the law of the formation of the indices of* v *in the expansion of* $(1+v)^m$: *and the coefficient of the second term.*

First, let the index be a positive whole number: then, since by actual multiplication, we have

$$(1+v)^2 = 1 + 2v + v^2:$$

$$(1+v)^3 = 1 + 3v + 3v^2 + v^3:$$

$$(1+v)^4 = 1 + 4v + 6v^2 + 4v^3 + v^4: \ \&c.$$

we perceive that the indices of v increase regularly by 1 in each succeeding term, and that the coefficient of the second term is the index of the binomial: whence assuming, in accordance with this observation, that

$$(1+v)^{m-1} = 1 + (m-1)v + B_1v^2 + B_2v^3 + \&c.:$$

we shall have

$$\begin{aligned}(1+v)^m &= (1+v)(1+v)^{m-1}\\ &= (1+v)\left\{1 + (m-1)v + B_1v^2 + B_2v^3 + \&c.\right\}\\ &= 1 + (m-1)v + B_1v^2 + B_2v^3 + \&c.\\ &\qquad + v + (m-1)v^2 + B_1v^3 + \&c.\\ \hline &= 1 + mv + (B_1 + m - 1)v^2 + (B_2 + B_1)v^3 + \&c.:\end{aligned}$$

from which we infer, that if the indices in each succeeding term increase by 1, and the coefficient of the second term be equal to the index of the binomial, for any one value of that index, the same will hold good for the next superior index: but this has been proved true for the values 2, 3, 4 of m: therefore, it is true when $m = 5$: hence also, when $m = 6$, and

therefore, when $m = 7$, and so on: and consequently it is generally true, that

$$(1+v)^m = 1 + mv + Bv^2 + Cv^3 + \&c.$$

when the index is a positive whole number.

Secondly, let the index be a fraction, and be denoted by $\frac{p}{q}$: and assume,

$$(1+v)^{\frac{p}{q}} = 1 + Av + \&c.: \quad \therefore \quad (1+v)^p = (1 + Av + \&c.)^q:$$

wherefore, by the preceding case, we shall have

$$1 + pv + \&c. = 1 + qAv + \&c.:$$

and equating coefficients, we obtain

$$qA = p, \text{ and } \therefore \ A = \frac{p}{q}:$$

and the indices of v must obviously increase as before:

$$\text{whence, } (1+v)^{\frac{p}{q}} = 1 + \frac{p}{q}v + Bv^2 + Cv^3 + \&c.$$

Thirdly, let the index be a negative quantity, either integral or fractional, represented by $-r$: then we have, by actual division,

$$(1+v)^{-r} = \frac{1}{(1+v)^r} = \frac{1}{1 + rv + \&c.} = 1 - rv + \&c.:$$

and the indices increase as before.

Hence, whether the index m be positive or negative, integral or fractional, the coefficient of the second term is m, and the form of the expansion will be

$$(1+v)^m = 1 + mv + Bv^2 + Cv^3 + \&c.$$

245. *To determine the law of the formation of the coefficients of the powers of* v, *in the expansion of* $(1+\text{v})^{\text{m}}$.

Let $(1+v)^m = 1 + mv + Bv^2 + Cv^3 + \&c.$, and for v put $y + z$: then, we shall have

$$(1 + y + z)^m = 1 + m(y + z) + B(y + z)^2 + C(y + z)^3 + \&c.$$
$$= 1 + my + By^2 + Cy^3 + Dy^4 + \&c.$$
$$+ mz + 2Byz + 3Cy^2z + 4Dy^3z + \&c.$$
$$+ \&c. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

by the preceding article:

again, by separating it into factors, we shall have

$$(1 + y + z)^m = \left\{(1 + y)\left(1 + \frac{z}{1 + y}\right)\right\}^m = (1 + y)^m\left(1 + \frac{z}{1 + y}\right)^m$$

$$= (1 + y)^m \left\{1 + m\left(\frac{z}{1 + y}\right) + B\left(\frac{z}{1 + y}\right)^2 + C\left(\frac{z}{1 + y}\right)^3 + \&c.\right\}$$

$$= (1 + y)^m + m(1 + y)^{m-1}z + B(1 + y)^{m-2}z^2 + C(1 + y)^{m-3}z^3 + \&c.$$
$$= 1 + my + By^2 + Cy^3 + Dy^4 + \&c.$$
$$+ mz\left\{1 + (m - 1)y + B'y^2 + C'y^3 + D'y^4 + \&c.\right\}$$
$$+ \&c. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

where B', C', D', &c. are the values of B, C, D, &c. when m is changed into $m - 1$, by the preceding article:

whence, by reason of the identity of these two expressions for $(1 + y + z)^m$, we must have the coefficients of the same symbols equal to each other in both: that is,

$$2B = m(m - 1), \text{ and } B = \frac{m(m - 1)}{1 \,.\, 2}:$$

$$\text{hence also, } B' = \frac{(m - 1)(m - 2)}{1 \,.\, 2}:$$

$$3C = mB' = \frac{m(m - 1)(m - 2)}{1 \,.\, 2}, \text{ and } C = \frac{m(m - 1)(m - 2)}{1 \,.\, 2 \,.\, 3}:$$

$$\text{hence likewise, } C' = \frac{(m - 1)(m - 2)(m - 3)}{1 \,.\, 2 \,.\, 3}:$$

$$4D = mC' = \frac{m(m - 1)(m - 2)(m - 3)}{1 \,.\, 2 \,.\, 3}:$$

$$\text{and } D = \frac{m(m - 1)(m - 2)(m - 3)}{1 \,.\, 2 \,.\, 3 \,.\, 4}:$$

and it is evident that this process may be continued as far as we please, so that

$$(1+v)^m = 1 + mv + \frac{m(m-1)}{1.2}v^2 + \frac{m(m-1)(m-2)}{1.2.3}v^3 + \frac{m(m-1)(m-2)(m-3)}{1.2.3.4}v^4 + \&c.$$

246. Cor. 1. By substituting $\frac{x}{a}$ in the place of v, we shall have

$$(a+x)^m = a^m\left(1+\frac{x}{a}\right)^m$$

$$= a^m\left\{1 + m\frac{x}{a} + \frac{m(m-1)}{1.2}\left(\frac{x}{a}\right)^2 + \frac{m(m-1)(m-2)}{1.2.3}\left(\frac{x}{a}\right)^3 + \&c.\right\}$$

$$= a^m + ma^{m-1}x + \frac{m(m-1)}{1.2}a^{m-2}x^2 + \frac{m(m-1)(m-2)}{1.2.3}a^{m-3}x^3 + \&c.$$

If m and x be both positive,

$$(a+x)^m = a^m + ma^{m-1}x + \frac{m(m-1)}{1.2}a^{m-2}x^2 + \&c.$$

If m be positive and x negative,

$$(a-x)^m = a^m - ma^{m-1}x + \frac{m(m-1)}{1.2}a^{m-2}x^2 - \&c.$$

If m be negative and x positive,

$$(a+x)^{-m} = a^{-m} - ma^{-m-1}x + \frac{m(m+1)}{1.2}a^{-m-2}x^2 - \&c.$$

$$= \frac{1}{a^m} - \frac{mx}{a^{m+1}} + \frac{m(m+1)x^2}{1.2a^{m+2}} - \&c.$$

If m and x be both negative,

$$(a-x)^{-m} = a^{-m} + ma^{-m-1}x + \frac{m(m+1)}{1.2}a^{-m-2}x^2 + \&c.$$

$$= \frac{1}{a^m} + \frac{mx}{a^{m+1}} + \frac{m(m+1)x^2}{1.2a^{m+2}} + \&c.$$

Also, if the index be fractional and be denoted by $\pm \frac{p}{q}$,

$$(a \pm x)^{\frac{p}{q}} = a^{\frac{p}{q}} \pm \frac{p}{q} a^{\frac{p}{q}-1} x + \frac{p(p-q)}{1 \cdot 2q^2} a^{\frac{p}{q}-2} x^2 \pm \&c.:$$

$$(a \pm x)^{-\frac{p}{q}} = a^{-\frac{p}{q}} \mp \frac{p}{q} a^{-\frac{p}{q}-1} x + \frac{p(p+q)}{1 \cdot 2q^2} a^{-\frac{p}{q}-2} x^2 \mp \&c.$$

247. Cor. 2. If t_1, t_2, t_3, &c. represent the first, second, third, &c. terms of the expansion, we may exhibit the theorem in another form.

Thus,

$$(a+x)^m = a^m + m\frac{x}{a}t_1 + \frac{m-1}{2}\frac{x}{a}t_2 + \frac{m-2}{3}\frac{x}{a}t_3 + \&c.:$$

by means of which, any term may easily be derived from that which immediately precedes it.

Ex. Required the fifth power of $2x + 3y$.

Here, $(2x+3y)^5 = (2x)^5 \left\{1 + \frac{3y}{2x}\right\}^5$: and by substituting in the general formula, 5 and $\frac{3y}{2x}$ in the places of m and v respectively, we have $(2x+3y)^5$

$$= 32x^5 \left\{1 + 5\left(\frac{3y}{2x}\right) + 10\left(\frac{3y}{2x}\right)^2 + 10\left(\frac{3y}{2x}\right)^3 + 5\left(\frac{3y}{2x}\right)^4 + \left(\frac{3y}{2x}\right)^5\right\}$$

$$= 32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3 + 810xy^4 + 243y^5.$$

248. *To find an expression for the* r^th^ *term, or the general term of the expansion of* $(1+v)^m$.

The first term is always $= 1$, and the second term $= mv$, and the general term will be determined by induction: for we have seen that

$$\text{the third term} = \frac{m(m-1)}{1 \cdot 2} v^2:$$

$$\text{the fourth term} = \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} v^3: \ \&c.$$

whence, observing the connection subsisting between the index of v and the factors of its coefficient, we shall have

$$\text{the } r^{\text{th}} \text{ term} = \frac{m\,(m-1)\,(m-2)\,\&\text{c.}\,(m-r+2)}{1\,.\,2\,.\,3\,.\,\&\text{c.}\,(r-1)}\,v^{r-1};$$

also, the coefficient of v^r, or of the $(r+1)^{\text{th}}$ term, will be

$$\frac{m\,(m-1)\,(m-2)\,\&\text{c.}\,(m-r+1)}{1\,.\,2\,.\,3\,.\,\&\text{c.}\,r},$$

which, by article (223), is the number of combinations of m things taken r together, when m is a positive whole number.

249. Cor. 1. If m be a positive whole number, and we suppose $m - r + 2 = 0$, or $r = m + 2$: the r^{th} and every succeeding term, involving zero as a factor, becomes $= 0$, and therefore the series terminates after the $(r-1)^{\text{th}}$, or $(m+1)^{\text{th}}$ term: that is, the expansion of a binomial, whose index is the positive integer m, contains $m + 1$ terms: and the number of terms of the expansion will therefore be even or odd, according as the index is odd or even.

Also, if m be either negative or fractional, it is manifest that no one of the factors of the r^{th} term can ever become $= 0$: and consequently the expansion will consist of an indefinite number of terms, and the developement admits only of a symbolical interpretation.

250. Cor. 2. If m be an even number, the middle terr of the expansion of $(1 + v)^m$ will be

$$\frac{\overset{1\,.\,3\,.\,5}{\cancel{1\,.\,2\,.\,3}}\,.\,\&\text{c.}\,(m-1)}{1\,.\,2\,.\,3\,.\,\&\text{c.}\,\frac{1}{2}m}\,(2v)^{\frac{m}{2}}.$$

For, the middle term, which is the $(\frac{1}{2}m+1)^{\text{th}}$, will

$$\frac{m\,(m-1)\,(m-2)\,\&\text{c.}\,(\frac{1}{2}m+1)}{1\,.\,2\,.\,3\,.\,\&\text{c.}\,\frac{1}{2}m}\,v^{\frac{m}{2}}$$ enough.

(which by multiplying both the terms by $1\,.\,2\,.\,3\,.\,\&\text{c.}\,\frac{1}{2}m$

$$= \frac{1\,.\,2\,.\,3\,.\,\&\text{c.}\,\frac{1}{2}m\,(\frac{1}{2}m+1)\,\&\text{c.}\,(m-2)\,(m-1)\,m}{(1\,.\,2\,.\,3\,.\,\&\text{c.}\,\frac{1}{2}m)^2},$$

$$= \frac{\{1.3.5.\&c.(m-1)\} \times \{2.4.6.\&c.\,m\}}{(1.2.3.\&c.\frac{1}{2}m)^2} v^{\frac{m}{2}}$$

$$= \frac{\{1.3.5.\&c.(m-1)\} \times \{1.2.3.\&c.\frac{1}{2}m\} 2^{\frac{m}{2}}}{(1.2.3.\&c.\frac{1}{2}m)^2} v^{\frac{m}{2}}$$

$$= \frac{1.3.5.\&c.(m-1)}{1.2.3.\&c.\frac{1}{2}m} (2v)^{\frac{m}{2}}.$$

251. Cor. 3. If m be an odd number, there will manifestly be two middle terms, the $\frac{1}{2}(m+1)^{\text{th}}$ and $\frac{1}{2}(m+3)^{\text{th}}$ from the beginning: and these will be found to be respectively equal to

$$\frac{m(m-1)(m-2)\&c.\frac{1}{2}(m+3)}{1.2.3.\&c.\frac{1}{2}(m-1)} v^{\frac{m-1}{2}},$$

$$\text{and} \quad \frac{m(m-1)(m-2)\&c.\frac{1}{2}(m+1)}{1.2.3.\&c.\frac{1}{2}(m+1)} v^{\frac{m+1}{2}}:$$

which, as in the last corollary, are easily made to assume the forms

$$\frac{1.3.5.\&c.m}{1.2.3.\&c.\frac{1}{2}(m+1)} 2^{\frac{m-1}{2}} v^{\frac{m-1}{2}} \text{ and } \frac{1.3.5.\&c.m}{1.2.3.\&c.\frac{1}{2}(m+1)} 2^{\frac{m-1}{2}} v^{\frac{m+1}{2}},$$

the coefficients of the powers of v being equal, agreeably to the purport of article (248).

Ex. 1. Required the general term of the expansion of

$$\frac{1}{\sqrt{1-x}} \text{ or } (1-x)^{-\frac{1}{2}}.$$

Here, $m = -\frac{1}{2}$, and $v = -x$; whence by substitution, we have

$$\text{the } r^{\text{th}} \text{ term} = \frac{m(m-1)(m-2)\&c.(m-r+2)}{1.2.3.\&c.(r-1)} v^{r-1}$$

$$= \frac{(-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)\&c.(-\frac{1}{2}-r+2)}{1.2.3.\&c.(r-1)} (-x)^{r-1}$$

$$= \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\,\&c.\left\{-\frac{1}{2}(2r-3)\right\}}{1\,.\,2\,.\,3\,.\,\&c.\,(r-1)}(-x)^{r-1}$$

$$= (-\tfrac{1}{2})^{r-1}\left\{\frac{1\,.\,3\,.\,5\,.\,\&c.\,(2r-3)}{1\,.\,2\,.\,3\,.\,\&c.\,(r-1)}\right\}(-x)^{r-1}$$

$$= \frac{1\,.\,3\,.\,5\,.\,\&c.\,(2r-3)}{1\,.\,2\,.\,3\,.\,\&c.\,(r-1)}\left(\frac{x}{2}\right)^{r-1}:$$

and this being positive, leads us to conclude that every term of the expansion will be positive also.

The first and second terms are 1 and $\frac{1}{2}x$:

if $r = 3$, the third term $= \dfrac{1\,.\,3}{1\,.\,2}\left(\dfrac{x}{2}\right)^2 = \dfrac{3}{8}x^2$:

if $r = 4$, the fourth term $= \dfrac{1\,.\,3\,.\,5}{1\,.\,2\,.\,3}\left(\dfrac{x}{2}\right)^3 = \dfrac{5}{16}x^3$:

if $r = 5$, the fifth term $= \dfrac{1\,.\,3\,.\,5\,.\,7}{1\,.\,2\,.\,3\,.\,4}\left(\dfrac{x}{2}\right)^4 = \dfrac{35}{128}x^4$: &c.

$$\therefore\ (1-x)^{-\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \frac{35}{128}x^4 + \&c.$$

Ex. 2. Find the r^{th} term of the expansion of $(a+x)^{\frac{1}{3}}$.

Here, $(a+x)^{\frac{1}{3}} = a^{\frac{1}{3}}\left(1+\dfrac{x}{a}\right)^{\frac{1}{3}}$: and therefore the general term of the expansion of $(a+x)^{\frac{1}{3}} = a^{\frac{1}{3}} \times$ the general term of the expansion of $\left(1+\dfrac{x}{a}\right)^{\frac{1}{3}}$:

also, putting $m = \frac{1}{3}$, and $v = \dfrac{x}{a}$, we shall have

the r^{th} term $= \dfrac{(\frac{1}{3})(\frac{1}{3}-1)(\frac{1}{3}-2)\,\&c.\,(\frac{1}{3}-r+2)}{1\,.\,2\,.\,3\,.\,\&c.\,(r-1)}\left(\dfrac{x}{a}\right)^{r-1}$

$$=\frac{(\frac{1}{3})(-\frac{2}{3})(-\frac{5}{3})\,\&c.\{-\frac{1}{3}(3r-7)\}}{1.2.3.\,\&c.\,(r-1)}\left(\frac{x}{a}\right)^{r-1}$$

$$=-\frac{(-\frac{1}{3})(-\frac{2}{3})(-\frac{5}{3})\,\&c.\{-\frac{1}{3}(3r-7)\}}{1.2.3.\,\&c.\,(r-1)}\left(\frac{x}{a}\right)^{r-1}$$

$$=-\left(-\frac{1}{3}\right)^{r-1}\left\{\frac{1.2.5.\,\&c.(3r-7)}{1.2.3.\,\&c.\,(r-1)}\right\}\left(\frac{x}{a}\right)^{r-1}$$

$$=-\frac{1.2.5.\,\&c.\,(3r-7)}{1.2.3.\,\&c.\,(r-1)}\left(-\frac{x}{3a}\right)^{r-1}:$$

which will be positive or negative, according as r is even or odd, and therefore the terms after the second will be alternately positive and negative.

The first and second terms of the expansion of $\left(1+\frac{x}{a}\right)^{\frac{1}{3}}$ are 1 and $\frac{x}{3a}$: also, by means of the general term,

if $r=3$, the third term $=-\frac{1.2}{1.2}\left(-\frac{x}{3a}\right)^2=-\frac{1}{9}\frac{x^2}{a^2}$:

if $r=4$, the fourth term $=-\frac{1.2.5}{1.2.3}\left(-\frac{x}{3a}\right)^3=\frac{5}{81}\frac{x^3}{a^3}$:

if $r=5$, the fifth term $=-\frac{1.2.5.8}{1.2.3.4}\left(-\frac{x}{3a}\right)^4=-\frac{10}{243}\frac{x^4}{a^4}$: &c.

whence, the developement of $(a+x)^{\frac{1}{3}}$ will be

$$a^{\frac{1}{3}}\left\{1+\frac{1}{3}\frac{x}{a}-\frac{1}{9}\frac{x^2}{a^2}+\frac{5}{81}\frac{x^3}{a^3}-\frac{10}{243}\frac{x^4}{a^4}+\&c.\right\}$$

$$=a^{\frac{1}{3}}+\frac{1}{3}\frac{x}{a^{\frac{2}{3}}}-\frac{1}{9}\frac{x^2}{a^{\frac{5}{3}}}+\frac{5}{81}\frac{x^3}{a^{\frac{8}{3}}}-\frac{10}{243}\frac{x^4}{a^{\frac{11}{3}}}+\&c.$$

252. *If* m *be a positive whole number, all the coefficients of* $(1 + v)^m$, *will be integral quantities.*

Let B_r denote the coefficient of the r^{th} term of the expansion of $(1+v)^{m-1}$, C_r that of the r^{th} term of the expansion of $(1+v)^m$: then, we have

$$B_r = \frac{(m-1)(m-2)\ \&c.\ (m-r+1)}{1.2.3.\ \&c.\ (r-1)}$$

$$= \frac{(m-1)(m-2)\ \&c.\ (m-r+2)}{1.2.3.\ \&c.\ (r-2)}\left\{\frac{m}{r-1} - 1\right\}$$

$$= \frac{m(m-1)(m-2)\ \&c.\ (m-r+2)}{1.2.3.\ \&c.\ (r-1)} - \frac{(m-1)(m-2)\ \&c.\ (m-r+2)}{1.2.3.\ \&c.\ (r-2)}$$

$$= C_r - B_{r-1}:$$

$$\text{whence,}\quad C_r = B_{r-1} + B_r:$$

if, therefore, all the coefficients be whole numbers for any one positive integral value of the index, all the coefficients will be whole numbers for the next superior value of it: but when the index is 2, 3 or 4, the coefficients have been shewn to be integral, and therefore, by successive induction, they are proved to be always integral.

This appears also from the consideration that the expansion of $(a+x)^m$ being the continued product of m factors each equal to $x+a$, can admit no fractional coefficients into its terms.

253. *When the index is a positive whole number, the coefficient of any term of the expansion reckoned from the end, is the same as the coefficient of the corresponding term reckoned from the beginning.*

Since the number of terms of the expansion is $m+1$, the *first* term from the end is the $(m+1)^{th}$ term from the beginning: the *second* term from the end is the m^{th} term from the beginning: the *third* term from the end is the $(m-1)^{th}$ term from the beginning, &c.: from which we infer that the r^{th} term from the end is the $\{(m+1)-(r-1)\}^{th}$ term from the beginning: whence, by substituting $m-r+2$ in the place

of r in the general term, we shall have the coefficient of the r^{th} term from the end

$$= \frac{m(m-1)(m-2) \&c.\ r}{1.2.3. \&c.\ (m-r+1)}$$

$$= \frac{m(m-1)(m-2) \&c.\ r(r-1) \&c.\ 3.2.1}{\{1.2.3. \&c.\ (r-1)\}\ \{1.2.3. \&c.\ (m-r+1)\}}$$

$$= \frac{m(m-1)(m-2) \&c.\ (m-r+2)}{1.2.3. \&c.\ (r-1)} \times \frac{(m-r+1) \&c.\ 3.2.1}{1.2.3. \&c.\ (m-r+1)}$$

$$= \frac{m(m-1)(m-2) \&c.\ (m-r+2)}{1.2.3. \&c.\ (r-1)},$$

which is the coefficient of the r^{th} term from the beginning: that is, the coefficients of the expanded binomial, are

$$1,\ m,\ \frac{m(m-1)}{1.2},\ \&c.,\ \frac{m(m-1)}{1.2},\ m,\ 1.$$

The expansions of $(a+x)^m$ and $(x+a)^m$ being equal to each other, it is evident that the coefficients which do not depend upon the powers of a and x, will be the same in both series, and consequently that the coefficients equidistant from the ends will be identical.

254. *To find expressions for the sums of the terms in the odd and even places, of the expansion of* $(1+v)^m$, *when* m *is a positive whole number.*

Since, $(1+v)^m = 1 + mv + \dfrac{m(m-1)}{1.2} v^2 + \&c.$

and $(1-v)^m = 1 - mv + m\dfrac{(m-1)}{1.2} v^2 - \&c.$:

by addition, and division by 2, we obtain

$$\frac{(1+v)^m + (1-v)^m}{2}$$

$$= 1 + \frac{m(m-1)}{1.2} v^2 + \frac{m(m-1)(m-2)(m-3)}{1.2.3.4} v^4 + \&c.$$

= the sum of the terms in the odd places:

and by subtraction, and division by 2, we have

$$\frac{(1+v)^m - (1-v)^m}{2}$$

$$= mv + \frac{m(m-1)(m-2)}{1.2.3} v^3 + \&c.$$

which is the sum of the terms in the even places.

255. COR. Making $v=1$, we shall have the sum of the *coefficients* of the terms in the odd places

$$= \frac{2^m}{2} = 2^{m-1}$$

= the sum of the *coefficients* of the terms in the even places: whence, the sum of *all* the coefficients of an expanded binomial, whose index is the positive whole number m, will be equal to $2 \times 2^{m-1} = 2^m$.

These conclusions follow immediately from the expansions:

$$2^m = (1+1)^m = 1 + m + \frac{m(m-1)}{1.2} + \frac{m(m-1)(m-2)}{1.2.3} + \&c.$$

$$0 = (1-1)^m = 1 - m + \frac{m(m-1)}{1.2} - \frac{m(m-1)(m-2)}{1.2.3} + \&c.$$

256. *To find the greatest term of the expansion of* $(1+v)^m$: *or, the term at which the series becomes convergent.*

Let r denote the place of the required term, and suppose T_r and T_{r+1} to represent the r^{th} and $(r+1)^{th}$ terms of the expansion: then

$$\frac{T_{r+1}}{T_r} = \left(\frac{m-r+1}{r}\right) v:$$

whence, if T_{r+1} be not greater than T_r for any one value of r, it will manifestly be so for every succeeding value of r: and corresponding to the greatest term, we must therefore have

$$\left(\frac{m-r+1}{r}\right) v \text{ not greater than } 1:$$

$\therefore\ (m - r + 1)\,v$ is not greater than r:

$\therefore\ (m + 1)\,v$ is not greater than $(1 + v)\,r$:

$$\therefore\ (m + 1)\frac{v}{1+v} \text{ is not greater than } r:$$

and r must therefore be the whole number which is equal to, or next greater than $(m + 1)\dfrac{v}{1+v}$, according as this quantity is integral or fractional.

Also, if $v = 1$, the greatest coefficient will correspond to the value of r, which is equal to, or next greater than $\frac{1}{2}(m + 1)$, in the same circumstances. See article (226).

257. COR. Since $(a + x)^m = a^m\left(1 + \dfrac{x}{a}\right)^m$, the greatest term of the expansion of $(a + x)^m = a^m \times$ the greatest term of the expansion of $\left(1 + \dfrac{x}{a}\right)^m$: and the place of the greatest will therefore be found by taking r, equal to, or next greater than $(m + 1)\dfrac{x}{a + x}$: as appears from substituting $\dfrac{x}{a}$ in the place of v.

258. These two articles are of the greatest importance in approximating to the roots of numerical magnitudes, by means of the binomial theorem, inasmuch as they point out the term after which the succeeding terms become continually less and less, or the series becomes convergent: and the limit of the error occasioned by stopping at this term, may be found by means of article (206).

Ex. 1. To approximate to the square root of 10.

$$\text{The square root of } 10 = \sqrt{9 + 1} = 3\sqrt{1 + \frac{1}{9}}$$

$$= 3\left\{1 + \frac{1}{2}\left(\frac{1}{9}\right) - \frac{1}{2\,.\,4}\left(\frac{1}{9}\right)^2 + \frac{1}{2\,.\,8}\left(\frac{1}{9}\right)^3 - \&c.\right\}$$

$$= 3 + \frac{1}{2\,.\,3} - \frac{1}{2\,.\,4\,.\,27} + \frac{1}{2\,.\,8\,.\,243} - \&c.:$$

$$\text{also, } (m+1)\frac{v}{1+v} = \left(\frac{1}{2}+1\right)\frac{\frac{1}{9}}{1+\frac{1}{9}} = \frac{3}{2}\times\frac{1}{10} = \frac{3}{20},$$

which shews that the series begins to converge immediately, and a small number of terms will give a good approximation.

Ex. 2. Approximate to the fifth root of 260.

Here, $260 = 243 + 17 = 3^5 + 17$: $\therefore$ if $a = 3^5$, and $x = 17$, we shall have

$$\sqrt[5]{260} = (a+x)^{\frac{1}{5}}$$

$$= a^{\frac{1}{5}}\left\{1 + \frac{1}{5}\left(\frac{x}{a}\right) - \frac{1.4}{2.5^2}\left(\frac{x}{a}\right)^2 + \frac{1.4.9}{2.3.5^3}\left(\frac{x}{a}\right)^3 - \text{\&c.}\right\}:$$

now, of the quantities included between the brackets,

the first term $= 1 = t_1$:

the second term $= \frac{1}{5}\left(\frac{x}{a}\right) t_1 = .0139917$ &c. $= t_2$:

the third term $= \frac{2}{5}\left(\frac{x}{a}\right) t_2 = .0003915$ &c. $= t_3$:

the fourth term $= \frac{3}{5}\left(\frac{x}{a}\right) t_3 = .0000164$ &c. $= t_4$:

&c. = &c. = &c.

whence, by substitution in the expression above, we find

$$\sqrt[5]{260} = 3\,\{1.0136166 \text{ \&c.}\} = 3.0408498 \text{ \&c.}$$

Ex. 3. Find at what term, the expansion of $\left(1+\frac{5}{6}\right)^{\frac{3}{2}}$ becomes convergent.

Here, we have $(m+1)\frac{v}{1+v} = 2\frac{1}{2}$: and the third term is therefore the greatest: and by actual developement it will appear that the fourth term is $\frac{4375}{1152}$, which is less than the third

term found to be $\frac{175}{32}$: that is, the series begins to *converge* after the *third* term: also, since $\frac{4375}{1152} \div \frac{175}{32} = \frac{25}{36}$, is greater than $\frac{1}{2}$, it appears from article (206), that $\frac{175}{32}$ is less than the sum of all the terms that follow it: and the same article will prove that $\frac{4375}{1152}$ is greater than the sum of all the succeeding terms: and thus the limit of the error occasioned by stopping at this term is ascertained.

259. *To approximate to the roots of numerical magnitudes, without the use of series.*

Let a be an approximate value of the m^{th} root of N, such that $\sqrt[m]{N} = a + x$, x being small:

$$\therefore \; N = (a + x)^m = a^m + ma^{m-1}x + \frac{m(m-1)}{1 \cdot 2} a^{m-2}x^2 + \&c.:$$

whence, $N - a^m = ma^{m-1}x$, nearly, and $\therefore \; x = \dfrac{N - a^m}{ma^{m-1}}$, nearly:

$$\text{but } N - a^m = \left(ma^{m-1} + \frac{m(m-1)}{1 \cdot 2} a^{m-2}x\right) x, \text{ nearly,}$$

$$= \left\{ma^{m-1} + \frac{(m-1)(N - a^m)}{2a}\right\} x, \text{ nearly,}$$

$$= \left\{\frac{(m+1)a^m + (m-1)N}{2a}\right\} x, \text{ nearly:}$$

$$\therefore \; x = \frac{2a(N - a^m)}{(m+1)a^m + (m-1)N}, \text{ nearly:}$$

$$\text{whence, } \sqrt[m]{N} = a + x = a + \frac{2a(N - a^m)}{(m+1)a^m + (m-1)N}, \text{ nearly,}$$

$$= \left\{\frac{(m+1)N + (m-1)a^m}{(m-1)N + (m+1)a^m}\right\} a, \text{ nearly,}$$

which is a closer approximation to the true root: let this be called a': and by a repetition of the same process, we shall manifestly have

$$\sqrt[m]{N} = \left\{\frac{(m+1)N + (m-1)a'^m}{(m-1)N + (m+1)a'^m}\right\} a', \text{ nearly},$$

which is still nearer to the true value: and so on, to any required degree of nicety.

260. Cor. If $N = a^m \pm b$, we shall have by substitution,

$$\sqrt[m]{N} = a \pm \frac{2ab}{(m+1)a^m + (m-1)(a^m \pm b)}, \text{ nearly},$$

$$= a \pm \frac{2ab}{2ma^m \pm (m-1)b}, \text{ nearly},$$

as a first approximation; and if this approximate value be called a', and the process be repeated, we shall have

$$\sqrt[m]{N} = a' \pm \frac{2a'b'}{2ma'^m \pm (m-1)b'}, \text{ more nearly},$$

where $N = a'^m \pm b'$: and so on.

For the square root, we shall have

$$\sqrt{N} = \left\{\frac{3N + a^2}{N + 3a^2}\right\} a = a + \frac{2ab}{4a^2 + b}, \text{ nearly}:$$

for the cube root, we shall similarly have

$$\sqrt[3]{N} = \left\{\frac{4N + 2a^3}{2N + 4a^3}\right\} a = a + \frac{2ab}{6a^3 + 2b}, \text{ nearly}.$$

261. *If* $(a + \sqrt{b})^{\frac{1}{m}} = x + \sqrt{y}$, *then will*

$$(a - \sqrt{b})^{\frac{1}{m}} = x - \sqrt{y}.$$

For, $a + \sqrt{b} = (x + \sqrt{y})^m$

$$= x^m + mx^{m-1}\sqrt{y} + \frac{m(m-1)}{1 \cdot 2} x^{m-2} y + \&c.:$$

whence, equating the rational and surd terms, we have

$$a = x^m + \frac{m(m-1)}{1 \cdot 2} x^{m-2} y + \&c.:$$

$$\sqrt{b} = mx^{m-1}\sqrt{y} + \frac{m(m-1)(m-2)}{1.2.3}x^{m-3}y\sqrt{y} + \&c.:$$

$$\therefore\ a - \sqrt{b} = x^m - mx^{m-1}\sqrt{y} + \frac{m(m-1)}{1.2}x^{m-2}y - \&c.$$

$$= (x - \sqrt{y})^m:$$

$$\text{and } (a - \sqrt{b})^{\frac{1}{m}} = x - \sqrt{y}.$$

Here, it is understood that $\sqrt{b}$ and $\sqrt{y}$ involve the same irrational factor: and in the same manner, when m is an odd number, if $\sqrt{a}$ and $\sqrt{b}$ involve the same irrational factors as $\sqrt{x}$ and $\sqrt{y}$ respectively, and

$$(\sqrt{a} + \sqrt{b})^{\frac{1}{m}} = \sqrt{x} + \sqrt{y}:$$

$$\text{then will } (\sqrt{a} - \sqrt{b})^{\frac{1}{m}} = \sqrt{x} - \sqrt{y}.$$

262. *To extract the* m[th] *root of a binomial, one or both of whose terms are possible quadratic surds.*

Let the proposed surd be represented by $\alpha + \beta$, where α is greater than β: and assume

$$\sqrt[m]{(\alpha + \beta)q} = x + y:$$

$$\therefore\ \sqrt[m]{(\alpha - \beta)q} = x - y, \text{ by the last article}:$$

$$\text{whence, } \sqrt[m]{(\alpha^2 - \beta^2)q^2} = x^2 - y^2:$$

and if q be assumed such that $(\alpha^2 - \beta^2)q^2$ is a complete m^{th} power as p^m, we shall have $x^2 - y^2 = p$:

$$\text{also, } \sqrt[m]{(\alpha + \beta)^2 q^2} + \sqrt[m]{(\alpha - \beta)^2 q^2}$$

$$= (x + y)^2 + (x - y)^2 = 2(x^2 + y^2),$$

which is evidently integral:

whence, if r and s be approximate values of the quantities

$$\sqrt[m]{(\alpha+\beta)^2 q^2} \text{ and } \sqrt[m]{(\alpha-\beta)^2 q^2},$$

such that one of them is greater, and the other less than the true value, we shall have

$$x^2 + y^2 = \tfrac{1}{2}(r+s),$$

which, together with $x^2 - y^2 = p$, gives

$$x = \tfrac{1}{2}(r+s+2p)^{\frac{1}{2}}, \text{ and } y = \tfrac{1}{2}(r+s-2p)^{\frac{1}{2}}:$$

$$\text{and } \therefore\ \sqrt[m]{\alpha+\beta} = \frac{1}{2\sqrt[m]{q}}\{(r+s+2p)^{\frac{1}{2}} + (r+s-2p)^{\frac{1}{2}}\},$$

whenever the root can be extracted.

Ex. 1. Extract the cube root of $9 + 4\sqrt{5}$.

$$\text{Assume, } \sqrt[3]{(9+4\sqrt{5})\,q} = x + y:$$

$$\therefore\ \sqrt[3]{(9-4\sqrt{5})\,q} = x - y:$$

$$\text{whence, } \sqrt[3]{(81-80)\,q^2} = x^2 - y^2,\ q = 1 \text{ and } x^2 - y^2 = 1:$$

$$\text{also, } 2(x^2+y^2) = \sqrt[3]{(9+4\sqrt{5})^2} + \sqrt[3]{(9-4\sqrt{5})^2}$$

$$= \sqrt[3]{161+72\sqrt{5}} + \sqrt[3]{161-72\sqrt{5}}$$

$$= \sqrt[3]{321.992 \text{ \&c.}} + \sqrt[3]{0.008 \text{ \&c.}}$$

$$= 7 - \delta + 0 + \delta = 7:$$

$$\therefore\ x^2 + y^2 = \frac{7}{2}, \text{ and } x^2 - y^2 = 1:$$

$$\text{whence, } x = \frac{3}{2}, \text{ and } y = \frac{\sqrt{5}}{2}:$$

$$\therefore\ \sqrt[3]{9+4\sqrt{5}} = \frac{3+\sqrt{5}}{2}, \text{ as may be verified.}$$

Ex. 2. Required the fifth root of $41 - 29\sqrt{2}$.

$$\text{Assume } \sqrt[5]{(41-29\sqrt{2})\,q} = x - y:$$

$$\therefore\ \sqrt[5]{(41+29\sqrt{2})\,q} = x + y:$$

$$\text{whence, } \sqrt[5]{-q^2} = x^2 - y^2,\ q = -1, \text{ and } x^2 - y^2 = -1:$$

also, $2(x^2+y^2) = \sqrt[5]{(41-29\sqrt{2})^2} + \sqrt[5]{(41+29\sqrt{2})^2}$

$$= \sqrt[5]{3363 - 2378\sqrt{2}} + \sqrt[5]{3363 + 2378\sqrt{2}}$$

$$1 - \delta + 5 + \delta = 5 + \delta + 1 - \delta = 6:$$

$$\therefore\ x^2 - y^2 = -1, \text{ and } x^2 + y^2 = 3:$$

$$\text{whence, } x = 1, \text{ and } y = \sqrt{2}:$$

so that the required root $= \dfrac{1-\sqrt{2}}{-1} = \sqrt{2} - 1.$

In examples of this kind, $(\alpha^2 - \beta^2)\,q^2$ will always be made a perfect m^{th} power, by assuming $q^2 = (\alpha^2 - \beta^2)^{m-1}$: but a less value may generally be found to answer the purpose, by resolving $\alpha^2 - \beta^2$ into its prime factors, and then rendering each of them a complete m^{th} power.

Whenever an even root is to be extracted, it will be better to extract the square root as often as possible by means of article (107): and then the process of this article will render it of no importance whether α be greater or less than β, as appears in the second example.

263. Whatever real value be assigned to m,

$$(a + b\sqrt{-1})^m + (a - b\sqrt{-1})^m \text{ is real,}$$

$$\text{and } (a + b\sqrt{-1})^m - (a - b\sqrt{-1})^m \text{ is imaginary.}$$

For, by the binomial theorem, $(a \pm b\sqrt{-1})^m$

$$= a^m \pm m a^{m-1}(b\sqrt{-1}) + \frac{m(m-1)}{1\,.\,2}a^{m-2}(b\sqrt{-1})^2 \pm \text{ \&c.}$$

the terms of which are alternately real and imaginary:

whence, $(a + b\sqrt{-1})^m + (a - b\sqrt{-1})^m$

$$= 2\left\{a^m - \frac{m(m-1)}{1\,.\,2}a^{m-2}b^2 + \text{\&c.}\right\}, \text{ which is real:}$$

and, $(a + b\sqrt{-1})^m - (a - b\sqrt{-1})^m$

$$= 2\left\{m a^{m-1} b - \frac{m(m-1)(m-2)}{1\,.\,2\,.\,3}a^{m-3}b^3 + \text{\&c.}\right\}\sqrt{-1},$$

which is imaginary.

Hence, the quantity $\{a + b\sqrt{-1}\}^{\frac{1}{m}} + \{a - b\sqrt{-1}\}^{\frac{1}{m}}$, is always a real magnitude, though it is exhibited symbolically as an imaginary quantity.

264. *To extract the* m[th] *root of a binomial, one of whose terms is rational, and the other either irrational or imaginary, by the solution of an equation of* m *dimensions.*

Let $\alpha + \beta$ denote the proposed binomial, and assume

$$\sqrt[m]{(\alpha + \beta)\, q} = x + \sqrt{y}:$$

$$\therefore\ \sqrt[m]{(\alpha - \beta)\, q} = x - \sqrt{y}:$$

whence, we have $\sqrt[m]{(\alpha^2 - \beta^2)\, q^2} = x^2 - y$,

where q must be assumed such that $(\alpha^2 - \beta^2)\, q^2 = p^m$:

$$\therefore\ x^2 - y = p, \text{ and } y = x^2 - p:$$

$$\text{also, } (\alpha + \beta)\, q = x^m + m x^{m-1} \sqrt{y} + \frac{m\,(m-1)}{1\,.\,2} x^{m-2} y + \&\text{c.}:$$

whence, equating the rational parts, we have the equation

$$q\alpha = x^m + \frac{m\,(m-1)}{1\,.\,2} x^{m-2} (x^2 - p) + \&\text{c.}:$$

from which if the value of x be found, the value of $y = x^2 - p$ will become known.

Ex. Find the cube root of $-11 - 2\sqrt{-1}$.

Here, $\alpha^2 - \beta^2 = 121 + 4 = 125 = 5^3$, $\therefore\ q = 1$ and $p = 5$:

$$\text{whence, } y = x^2 - 5:$$

$$\text{also, } -11 = 4x^3 - 15x, \text{ which gives } x = 1:$$

$$\text{therefore } y = -4, \text{ or } \sqrt{y} = 2\sqrt{-1}:$$

and the required cube root $= 1 + 2\sqrt{-1}$.

265. By means of the expansion of $(1 - v)^{-m}$

$$= 1 + mv + \frac{m\,(m+1)}{1\,.\,2} v^2 + \&\text{c.},$$

the expansion of $(a + x)^m$ may be made to assume various different forms.

Since, $\frac{a+x}{a} = \frac{a+x}{a+x-x} = \frac{1}{1-\frac{x}{a+x}}$, we shall have

$$(a+x)^m = a^m\left(1-\frac{x}{a+x}\right)^{-m} = a^m\left\{1 + m\left(\frac{x}{a+x}\right)\right.$$

$$\left. + \frac{m(m+1)}{1.2}\left(\frac{x}{a+x}\right)^2 + \frac{m(m+1)(m+2)}{1.2.3}\left(\frac{x}{a+x}\right)^3 + \&c.\right\}.$$

Since, $\frac{a+x}{x} = \frac{a+x}{a+x-a} = \frac{1}{1-\frac{a}{a+x}}$, we shall have

$$(a+x)^m = a^m\left\{1 + m\left(\frac{a}{a+x}\right) + \frac{m(m+1)}{1.2}\left(\frac{a}{a+x}\right)^2 + \&c.\right\}.$$

Since, $\frac{a+x}{2x} = \frac{a+x}{(a+x)-(a-x)} = \frac{1}{1-\frac{a-x}{a+x}}$, we shall have

$$(a+x)^m = 2^m x^m\left\{1 + m\left(\frac{a-x}{a+x}\right) + \frac{m(m+1)}{1.2}\left(\frac{a-x}{a+x}\right)^2 + \&c.\right\}.$$

Since, $\frac{a+x}{2a} = \frac{a+x}{(a+x)+(a-x)} = \frac{1}{1+\frac{a-x}{a+x}}$, we shall have

$$(a+x)^m = 2^m a^m\left\{1 - m\left(\frac{a-x}{a+x}\right) + \frac{m(m+1)}{1.2}\left(\frac{a-x}{a+x}\right)^2 - \&c.\right\}.$$

These series, which are merely symbolical when m is a positive whole number, all immediately converge, and may be used with advantage whenever the numerators of the terms are small compared with the denominators.

266. The expansions of trinomials, quadrinomials, &c may be obtained from the Binomial Theorem, by consider two or more of their terms as one: thus,

$$(a+b+c)^m = \{a+(b+c)\}^m$$

$$= a^m + ma^{m-1}(b+c) + \frac{m(m-1)}{1\,.\,2}a^{m-2}(b+c)^2 + \&c.:$$

$$(a+b+c+d)^m = \{(a+b)+(c+d)\}^m = (a+b)^m$$

$$+ m(a+b)^{m-1}(c+d) + \frac{m(m-1)}{1.2}(a+b)^{m-2}(c+d)^2 + \&c.:$$

and so on: in each of which the developements indicated, must be effected, and the terms collected and arranged according to the dimensions of one of the letters involved.

Ex. To find the coefficient of x^r in the expansion of $(1+x+x^2)^m$.

Here, $(1+x+x^2)^m = \{1+x(1+x)\}^m$

$$= 1 + mx(1+x) + \frac{m(m-1)}{1\,.\,2}x^2(1+x)^2 + \&c.$$

$$+ \frac{m(m-1)\,\&c.\,(m-r+3)}{1\,.\,2\,.\,3\,.\,\&c.\,(r-2)}x^{r-2}(1+x)^{r-2}$$

$$+ \frac{m(m-1)\,\&c.\,(m-r+2)}{1\,.\,2\,.\,3\,.\,\&c.\,(r-1)}x^{r-1}(1+x)^{r-1}$$

$$+ \frac{m(m-1)\,\&c.\,(m-r+1)}{1\,.\,2\,.\,3\,.\,\&c.\,r}x^r(1+x)^r + \&c.:$$

and since every term beyond the $(r+1)^{\text{th}}$, contains a higher power of x than x^r as a factor, we need only find what parts of the preceding terms comprise x^r: and

the part of the $(r+1)^{\text{th}}$ term comprising x^r

$$= \frac{m(m-1)\,\&c.\,(m-r+1)}{1\,.\,2\,.\,3\,.\,\&c.\,r}x^r:$$

the part of the r^{th} term comprising x^r

th[illegible]t di[illegible]rent $= \dfrac{m(m-1)\,\&c.\,(m-r+2)}{1\,.\,2\,.\,3\,.\,\&c.\,(r-1)}\cdot\dfrac{(r-1)}{1}x^r:$

the part of the $(r-1)^{\text{th}}$ term comprising x^r

$$= \frac{m(m-1)\,\&c.\,(m-r+3)}{1.2.3.\&c.\,(r-2)} \cdot \frac{(r-2)(r-3)}{1.2} x^r : \ \&c. :$$

and the term required will be the sum of these.

Thus, to find the coefficient of x^4 in the expansion of $(1+x+x^2)^4$, we have

from the fifth term, $\dfrac{4.3.2.1}{1\ 2.3.4} x^4 = x^4$:

from the fourth term, $\dfrac{4.3.2}{1.2.3} \cdot \dfrac{3}{1} x^4 = 12x^4$:

from the third term, $\dfrac{4.3}{1.2} \cdot \dfrac{2.1}{1.2} x^4 = 6x^4$:

and the entire term $= x^4 + 12x^4 + 6x^4 = 19x^4$, as may easily be proved by actual multiplication.

These processes may be superseded by the use of the *Multinomial Theorem*, of which an account will be given in the first Appendix.

CHAPTER XI.

LOGARITHMS, INTEREST AND ANNUITIES.

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267. DEF. THE Logarithm of a number y, is such a value of the index x, of a fixed magnitude a, as will satisfy the equation $y = a^x$: that is, x is *defined* to be the logarithm of y in a *system of logarithms* whose *base* is a: and the logarithm of y will therefore depend entirely upon the quantity a, which may be assumed of any finite magnitude whatever, *unity* only excepted, because every arithmetical power or root of 1 is 1.

268. COR. Since $1 = a^0$, we have $0 =$ the logarithm of 1, in every system of logarithms.

Also, because $a = a^1$, it follows that the logarithm of the base of the system is always $= 1$.

Again, since $0 = \frac{1}{a^\infty} = a^{-\infty}$, it appears that the logarithm of 0 is an infinite negative quantity.

And, because the equation $-y = a^x$, or $y = -a^x$, cannot be satisfied by any value of x, whether positive or negative, the logarithms of negative quantities, in a system of logarithms whose base is a *real* magnitude, can have no existence, or are *imaginary*.

269. If the number 10, which is the base of the common system of notation, be adopted for the base of the logarithms as above defined, and the word logarithm be written in the abbreviated form *log.*, we have

$$1 = 10^0, \text{ or } \log 1 = 0:$$

$$10 = 10^1, \text{ or } \log 10 = 1:$$

$$100 = 10^2, \text{ or } \log 100 = 2:$$

$$1000 = 10^3, \text{ or } \log 1000 = 3: \text{ \&c.}$$

$$\frac{1}{10} = 10^{-1}, \text{ or } \log \frac{1}{10} = -1:$$

$$\frac{1}{100} = 10^{-2}, \text{ or } \log \frac{1}{100} = -2:$$

$$\frac{1}{1000} = 10^{-3}, \text{ or } \log \frac{1}{1000} = -3: \text{ \&c.}$$

Whence, it is easily seen, that the logarithm of any magnitude between 1 and 10, will be a fraction, which is usually expressed *decimally:* that of any magnitude between 10 and 100 will be 1, with a decimal fraction annexed: that of one between 100 and 1000, will be 2, with a corresponding decimal fraction: and so on. In the same manner, the logarithm of any magnitude between 1 and $\frac{1}{10}$ will be a negative quantity between 0 and -1, that of any magnitude between $\frac{1}{10}$ and $\frac{1}{100}$, will be a negative quantity between -1 and -2: and so on.

270. Def. The integers, 0, 1, 2, 3, &c. to the left of the decimal points in the logarithms of magnitudes, are called the *characteristics* of those logarithms, and the fractional portions, expressed decimally, are termed the *mantissæ.*

271. If the values of x in the equation $y = 10^x$, calculated for successive values of y from 1 upwards, be *registered*, or put in the form of a *table*, these values are called the tabular logarithms of the corresponding numbers in the *common* system of logarithms.

In *Babbage's* tables, which are the most correct, and best adapted to practical purposes, the mantissæ alone of all num-

bers from 1 to 108000, calculated to *seven* places of decimals, are inserted: and the characteristics are immediately supplied from the considerations mentioned in article (269).

272. Def. The principal object of the invention of logarithms is to facilitate the performance of the arithmetical operations of *Multiplication*, *Division*, *Involution*, and *Evolution;* more particularly in cases where the quantities employed consist of several figures, and near approximations to the true results are considered sufficient for the practical purposes to which they are applied: and how this is done will now be shewn.

$$(1) \qquad \operatorname{Log} y_1 y_2 = \log y_1 + \log y_2.$$

For, let $x_1 = \log y_1$, and $x_2 = \log y_2$ in a system of logarithms whose base is a: then, by the definition, we have

$$y_1 = a^{x_1}, \text{ and } y_2 = a^{x_2}; \text{ whence, } y_1 y_2 = a^{x_1 + x_2}:$$

$$\text{and } \therefore \log y_1 y_2 = x_1 + x_2 = \log y_1 + \log y_2.$$

Similarly, $\log y_1 y_2 y_3 \text{ \&c.} = \log y_1 + \log y_2 + \log y_3 + \text{\&c.}$

That is, the logarithm of a product, or of a composite number, is equal to the sum of the logarithms of all its factors: and *vice versâ.*

Thus, with the help of tables, the operation of *Multiplication* is reduced to that of *Addition:* and the logarithms of *composite* numbers are derived from those of their *factors.*

$$(2) \qquad \operatorname{Log} \frac{y_1}{y_2} = \log y_1 - \log y_2.$$

For, retaining the same notation, we have

$$\frac{y_1}{y_2} = \frac{a^{x_1}}{a^{x_2}} = a^{x_1 - x_2}:$$

$$\text{and } \therefore \log \frac{y_1}{y_2} = x_1 - x_2 = \log y_1 - \log y_2.$$

Similarly, $\log \dfrac{y_1 y_2 \,\&c.}{y' y'' \,\&c.} = \log y_1 y_2 \,\&c. - \log y' y'' \,\&c.$

$$= \log y_1 + \log y_2 + \&c. - \log y' - \log y'' - \&c.$$

That is, the logarithm of a fraction is equal to the logarithm of its numerator, diminished by the logarithm of its denominator: and the operation of *Division* is hereby reduced to that of *Subtraction.*

Hence it follows, that the logarithm of a *proper* fraction is *negative*, whilst that of an *improper* fraction is *positive.*

$$(3) \qquad \text{Log}\, y_1^m = m \log y_1.$$

As before, $y_1 = a^{x_1}$, and $\therefore\ y_1^m = (a^{x_1})^m = a^{mx_1}$:

$$\text{whence, } \log y_1^m = m x_1 = m \log y_1.$$

That is, the logarithm of any power of a quantity is found by multiplying the logarithm of the quantity by the index expressing the power; and thus, the operation of *Involution* is reduced to that of *Multiplication.*

$$(4) \qquad \text{Log}\, y_1^{\frac{1}{m}} = \frac{1}{m} \log y_1.$$

From $y_1 = a^{x_1}$, we have $y_1^{\frac{1}{m}} = (a^{x_1})^{\frac{1}{m}} = a^{\frac{x_1}{m}}$:

$$\text{and } \therefore\ \log y_1^{\frac{1}{m}} = \frac{1}{m} x_1 = \frac{1}{m} \log y_1.$$

That is, the logarithm of any root of a quantity is obtained by dividing the logarithm of the quantity by the number which indicates the root: and the operation of *Evolution* is thus reduced to that of *Division.*

These fundamental properties of logarithms having been established without reference to any particular value of a, will of course hold good for the tabular logarithms, whose base is 10.

EXAMPLES.

(1) Given $\log 2 = .3010300$ and $\log 3 = .4771213$, to find $\log 216$.

Here, $216 = 6^3 = 2^3 . 3^3$: whence we have

$$\log 216 = \log 2^3 . 3^3 = \log 2^3 + \log 3^3$$

$$= 3 \log 2 + 3 \log 3 = 3 (\log 2 + \log 3)$$

$$= 3(.3010300 + .4771213) = 3(.7781513) = 2.3344539.$$

(2) Given $\log 3 = .4771213$, $\log 1.38 = .1398791$ and $\log 6.240325 = .7952071$, to find the value of $\sqrt{\dfrac{3\sqrt[3]{138}}{\sqrt[5]{.01}}}$.

Let $x = \sqrt{\dfrac{3\sqrt[3]{138}}{\sqrt[5]{.01}}}$: then $x^2 = \dfrac{3\sqrt[3]{138}}{\sqrt[5]{.01}}$:

whence, $2 \log x = \log 3 + \dfrac{1}{3} \log 138 - \dfrac{1}{5} \log .01$

$$= \log 3 + \frac{1}{3} \log (1.38 \times 100) - \frac{1}{5} \log (1 \div 100)$$

$$= \log 3 + \frac{1}{3} \{\log 1.38 + 2\} - \frac{1}{5} \{\log 1 - 2\}$$

$$= .4771213 + \frac{1}{3} (2.1398791) + .4$$

$$= 1.5904143:$$

$$\therefore \log x = .7952071, \text{ and } x = 6.240325.$$

(3) To prove that $\left\{7 \log \dfrac{16}{15} + 5 \log \dfrac{25}{24} + 3 \log \dfrac{81}{80}\right\} \times$

$$\left\{16 \log \frac{16}{15} + 12 \log \frac{25}{24} + 7 \log \frac{81}{80}\right\} = \log (5^{\log 2}).$$

$$\text{Here, } \log\frac{16}{15} = \log\frac{32}{30} = \log\frac{2^5}{3.10} = 5\log 2 - \log 3 - 1:$$

$$\therefore\ 7\log\frac{16}{15} = 35\log 2 - 7\log 3 - 7:$$

$$\log\frac{25}{24} = \log\frac{100}{96} = \log\frac{10^2}{2^5.3} = 2 - 5\log 2 - \log 3:$$

$$\therefore\ 5\log\frac{25}{24} = -25\log 2 - 5\log 3 + 10:$$

$$\log\frac{81}{80} = \log\frac{3^4}{2^3.10} = 4\log 3 - 3\log 2 - 1:$$

$$\therefore\ 3\log\frac{81}{80} = -9\log 2 + 12\log 3 - 3:$$

whence, by addition, we obtain

$$7\log\frac{16}{15} + 5\log\frac{25}{24} + 3\log\frac{81}{80} = \log 2:$$

similarly, $16\log\frac{16}{15} + 12\log\frac{25}{24} + 7\log\frac{81}{80} = 1 - \log 2:$

and therefore, the product of the two quantities proposed

$$= \log 2\,(1 - \log 2) = \log 2 \log 5 = \log\left(5^{\log 2}\right).$$

(4) $\quad \text{Log}\,(x+1) = \log x + 2\log\dfrac{2x+2}{2x+1} + \log\dfrac{(2x+1)^2}{(2x+1)^2 - 1}.$

$$\text{For, } \log\frac{(2x+1)^2}{(2x+1)^2 - 1} = \log\frac{(2x+1)^2}{4x^2 + 4x} = \log\frac{(2x+1)^2}{4x(x+1)}$$

$$= 2\log(2x+1) - \log(x+1) - 2\log 2 - \log x:$$

$$2\log\frac{2x+2}{2x+1} = 2\log\frac{2(x+1)}{2x+1}$$

$$= 2\log(x+1) - 2\log(2x+1) + 2\log 2:$$

$$\therefore\ \log x + 2\log\frac{2x+2}{2x+1} + \log\frac{(2x+1)^2}{(2x+1)^2 - 1} = \log(x+1).$$

(5) Solve the exponential equation $a^x - 8a^{-x} = 2$.

Here, multiplying every term by a^x and transposing, we have $a^{2x} - 2a^x = 8$, a quadratic form:

$$\text{whence, } a^{2x} - 2a^x + 1 = 9, \quad \text{and } \therefore\ a^x - 1 = \pm 3,$$

$$\text{which gives } a^x = 4, \quad \text{and } a^x = -2:$$

$$\therefore\ x \log a = 2 \log 2, \quad \text{and } x = \frac{2 \log 2}{\log a}:$$

$$\text{also, } x \log a = \log(-2), \quad \text{and } x = \frac{\log(-2)}{\log a}:$$

the former of which is real, and the latter imaginary.

(6) Given the equations $a^{px} b^{qy} = c$ and $d^{rx} e^{sy} = f$, to find the values of x and y.

Taking the logarithms of both members of each, we have

$$px \log a + qy \log b = \log c, \quad \text{and } rx \log d + sy \log e = \log f:$$

$$\text{or, } A_1 x + B_1 y = C_1, \text{ and } A_2 x + B_2 y = C_2:$$

$$\text{whence, } x = \frac{B_2 C_1 - B_1 C_2}{A_1 B_2 - A_2 B_1} = \frac{s \log c \log e - q \log b \log f}{ps \log a \log e - qr \log b \log d}:$$

$$\text{and, } y = \frac{A_1 C_2 - A_2 C_1}{A_1 B_2 - A_2 B_1} = \frac{p \log a \log f - r \log a \log d}{ps \log a \log e - qr \log b \log d}.$$

(7) In a geometrical progression, we have $l = ar^{n-1}$:

$$\text{whence, } \log l = \log a + (n-1) \log r:$$

$$\text{and } \therefore\ n = 1 + \frac{\log l - \log a}{\log r}, \text{ is found}:$$

$$\text{also, we have } s = \frac{a(r^n - 1)}{r - 1}, \text{ or } r^n = \frac{a + (r-1)s}{a}:$$

$$\text{whence, } n \log r = \log\{a + (r-1)s\} - \log a,$$

$$\text{and } \therefore\ n = \frac{\log\{a + (r-1)s\} - \log a}{\log r}, \text{ is determined.}$$

INTEREST.

273. Def. Interest is the *value* of, or the *consideration* paid for, the *use* of money, and the *rate* of interest is the sum paid for the use of a certain sum for a certain time, as of £1. for 1 year; or of £100. for 1 year, in which case it is termed the rate *per cent.*, *per annum*.

The *Principal* is the whole sum of money which produces interest.

The *Amount* is the sum of the principal and its interest for any time, taken together.

When the principal alone produces interest, it is called *Simple Interest:* but when the interest, as soon as it becomes due, is added to the principal, and the whole then produces interest, it is termed *Compound Interest.*

274. *To find expressions for the simple interest, and amount of a given sum, in a given time, at a given rate.*

Let p = the principal or sum lent:

r = the interest of £1. for 1 year:

n = the number of years:

i = the interest of the sum lent:

m = the amount of that sum:

then, rp = the interest of p£ for one year:

and, nrp = the interest of p£ for n years:

whence, we have $i = nrp$, the interest required:

also, $m = p + i = p + nrp = (1 + nr)\,p$, the amount sought.

275. The two formulæ above investigated,

$$(1) \quad i = nrp, \qquad (2) \quad m = (1 + nr)\,p,$$

will enable us to solve every question connected with this subject: and if any three of the quantities concerned be given,

the remaining one will be immediately found, by the solution of a simple equation.

276. Cor. If ρ be the rate per cent., it is evident that $\rho = 100r$, or $r = \frac{\rho}{100}$: and the equations will become,

$$i = \frac{n\rho p}{100}, \quad \text{and,} \quad m = \left(1 + \frac{n\rho}{100}\right)p = \left(\frac{100 + n\rho}{100}\right)p:$$

which, put into words, are the Arithmetical Rules commonly used.

Ex. 1. Required the simple interest, and amount of £125. 6*s*. 8*d*. in 4 years, at 5 per cent.

Here, $p = £125.\ 6s.\ 8d. = 125\frac{1}{3}$:

$$r = \frac{5}{100} = \frac{1}{20} = .05: \text{ and } n = 4:$$

$$\therefore\ i = 4 \times \frac{1}{20} \times 125\frac{1}{3} = 25\frac{1}{15} = £25.\ 1s.\ 4d.:$$

$$\text{also, } m = (1 + nr)p = \left\{1 + 4\left(\frac{1}{20}\right)\right\} 125\frac{1}{3} = \frac{6}{5} \times 125\frac{1}{3}$$

$$= \frac{752}{5} = £150.\ 8s.$$

Here also, $m = p + i = £125.\ 6s.\ 8d. + £25.\ 1s.\ 4d. = £150.\ 8s.$

Ex. 2. The interest of £25. for $3\frac{1}{2}$ years, at simple interest, was found to be £3. 18*s*. 9*d*.: required the rate per cent.

$$\text{From } i = nrp, \text{ we have } r = \frac{i}{np} = \frac{3.9375}{(3.5) \times 25}$$

$$= \frac{3.9375}{87.5} = .045:$$

whence, the rate per cent. $\rho = .045 \times 100 = 4.5 = 4\frac{1}{2}$.

Ex. 3. If p£ at simple interest, amount to m£ in t years: what sum must be paid down to receive p£ at the end of t years?

Let x be the required sum: then, if r be the rate of interest, we shall have

$$m = (1 + rt)p, \text{ and } p = (1 + rt)x;$$

$$\text{whence, } \frac{(1 + rt)x}{(1 + rt)p} = \frac{p}{m}, \text{ and } \therefore x = \frac{p^2}{m}.$$

Ex. 4. Each of two persons A and B bought £300. into the stocks, A into the three per cents. and B into the four per cents., and B receives £1. a year more interest than A: when both stocks had risen 10 per cent., they sold out, and A received £10. more than B: required the original prices of the stocks.

Let x = the price of £100. stock in the 3 per cents.:

y = ... 4 .... :

$$\text{then, } x : 300 :: 3 : \frac{900}{x} = A\text{'s annual interest}:$$

$$\text{and, } y : 300 :: 4 : \frac{1200}{y} = B\text{'s annual interest}:$$

$$\text{whence, } \frac{1200}{y} = \frac{900}{x} + 1, \text{ or } 1200x - 900y = xy:$$

also, $\dfrac{300}{x}$ and $\dfrac{300}{y}$ are as the quantities of stock possessed respectively by A and B, which they sell at $x + 10$ and $y + 10$:

$$\therefore \frac{300}{x}(x + 10) - \frac{300}{y}(y + 10) = 10, \text{ by the question}:$$

which reduced, gives $300(y - x) = xy$:

therefore, from the two equations,

$$1200x - 900y = xy, \text{ and } 300(y - x) = xy,$$

the values of x and y may be found: thus,

$$300y - 300x = 1200x - 900y: \therefore y = \frac{5}{4}x:$$

$$\text{whence, } \frac{5}{4}x^2 = 300(y - x) = 300\left(\frac{5}{4}x - x\right) = \frac{300x}{4}.$$

$$\therefore x = £60, \text{ the price of the 3 per-cents.:}$$

$$\text{and } y = \frac{5}{4}x = £75, \text{ the price of the 4 per-cents.}$$

277. DEF. *Discount* is an abatement on a sum of money, when payment is made before it becomes due: and if the discount be subtracted from the sum, the remainder is termed its *Present Worth.*

278. *To find expressions for the present worth and discount of a given sum, due at a given time, at a given rate.*

Let p denote the present worth of the sum m, due n years hence, at the rate r:

then it is manifest that in n years, p must amount to m at the given rate:

$$\text{whence, we shall have } m = (1 + nr)p:$$

$$\text{and therefore } p = \frac{m}{1 + nr}, \text{ the present worth:}$$

$$\text{also, the discount } d = m - \frac{m}{1 + nr} = \frac{m + nrm - m}{1 + nr} = \frac{nrm}{1 + nr}.$$

If ρ denote the rate per cent., we shall have

$$p = \frac{m}{1 + \frac{n\rho}{100}} = \frac{100m}{100 + n\rho}, \text{ and } d = \frac{nm\frac{\rho}{100}}{1 + \frac{n\rho}{100}} = \frac{n\rho m}{100 + n\rho}:$$

which, enunciated at length, are the rules used in common Arithmetic.

From these two equations, if any three of the quantities be given, the remaining one may be found: and since the interest of m in the time n, at the rate r is nrm, it is manifest that interest is always greater than discount under the same circumstances.

Ex. 1. Find the present worth, and discount of £440. 10s. due $1\frac{1}{4}$ years hence, at $3\frac{1}{2}$ per cent.

$$\text{Generally, } p = \frac{m}{1+nr}, \text{ and here } m = 440.5:$$

$$n = 1.25 \quad \text{and} \quad r = .035:$$

$$\text{whence, } p = \frac{440.5}{1 + (1.25)(.035)} = \frac{440.5}{1.04375}$$

$$= 422.035928 = £422.\ 0s.\ 8\tfrac{1}{2}d., \text{ nearly}:$$

and $d = £440.\ 10s. - £422.\ 0s.\ 8\frac{1}{2}d. = £18.\ 9s.\ 3\frac{1}{2}d.$, nearly.

Ex. 2. What sum must be paid down to receive £1000. 10s. at the end of 5 years 4 months, allowing interest at the rate of $4\frac{1}{2}$ per cent. per annum?

$$\text{Here, } m = 1000\tfrac{1}{2}, \quad n = 5\tfrac{1}{3}, \quad \text{and} \quad r = \frac{9}{200}:$$

$$\text{whence, } p = \frac{1000\frac{1}{2}}{1 + (5\frac{1}{3})\left(\frac{9}{200}\right)} = \frac{1000\frac{1}{2}}{1 + \frac{6}{25}} = \frac{50025}{62} = £806\tfrac{53}{62}.$$

Ex. 3. The interest of a sum of money for t years, is to the discount of the same sum payable at the end of t years, as h to k: find the rate per cent.

$$\text{Here, we have } \frac{h}{k} = rtm \div \frac{rtm}{1+rt} = 1 + rt:$$

$$\therefore\ rt = \frac{h-k}{k}, \text{ and the rate per cent.} = \frac{100\,(h-k)}{kt}.$$

279. DEF. The *Equation of Payments* is the finding a time, at which two or more sums of money due at different

periods, may be paid together without prejudice to the payer or receiver.

280. *To find the equated time of payment of two sums due at different periods, at a given rate of interest.*

Let S, s be the sums due at the ends of the times T, t: r the rate of interest: then, it is manifest that the present values of these sums, due at their respective times, should in equity be together equal to the present value of their sum, due at the equated time:

whence, if the equated time be denoted by x, we shall have

$$\text{the present value of } S = \frac{S}{1 + rT}:$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad s = \frac{s}{1 + rt}:$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots S + s = \frac{S + s}{1 + rx}:$$

$$\therefore \frac{S}{1 + rT} + \frac{s}{1 + rt} = \frac{S + s}{1 + rx}, \text{ determines the value of } x:$$

$$\text{thus, } S + rSt + s + rsT + rx(S + s + rSt + rsT)$$

$$= (1 + rT + rt + r^2 Tt)(S + s)$$

$$= S + rST + rSt + r^2 STt + s + rsT + rst + r^2 sTt:$$

$$\therefore rx(S + s + rSt + rsT) = r(ST + st + rSTt + rsTt):$$

$$\text{whence, } x = \frac{ST + st + r(S + s)Tt}{S + s + r(St + sT)}:$$

which is the correct value of the equated time.

If r be a very small quantity, as in practice it usually is, we shall have

$$x = \frac{ST + st}{S + s}, \text{ nearly}:$$

which gives the common approximate rule.

281. Cor. We have just seen that

$$x = \frac{ST + st}{S + s} \left\{ \frac{1 + \dfrac{r(S+s)Tt}{ST+st}}{1 + \dfrac{r(St+sT)}{S+s}} \right\}:$$

and the quantity between the brackets will be a proper or improper fraction, according as

$$\frac{r(S+s)Tt}{ST+st} < \text{ or } > \frac{r(St+sT)}{S+s}:$$

$$\text{as } (S+s)^2 Tt < \text{ or } > (ST+st)(St+sT):$$

$$\text{as } S^2Tt + 2SsTt + s^2Tt < \text{ or } > S^2Tt + SsT^2 + Sst^2 + s^2Tt:$$

$$\text{as } 2SsTt < \text{ or } > Ss(T^2 + t^2):$$

$$\text{as } 2Tt < \text{ or } > T^2 + t^2:$$

but, by article (49), $2Tt$ is less than $T^2 + t^2$: and therefore the quantity between the brackets is a proper fraction, and x is less than $\dfrac{ST+st}{S+s}$: and consequently the common approximate rule is in favour of the payer, inasmuch as the period of payment is deferred longer than it ought to be.

If the simple power of r be retained, a second approximation will be found to give

$$x = \frac{ST+st}{S+s} - \frac{Ss(T-t)^2}{(S+s)^2} r, \text{ nearly.}$$

If there be more sums than two, as S, s, σ due at the times T, t, τ, we have only to consider $S+s$ as due at the time x, and σ at the time τ, and to proceed as before, both for the true and the approximate equated time.

282. *To find the amount, and compound interest, of a given sum, in a given time, at a given rate.*

Let the interest be converted into principal at the end of every year, and a similar notation be retained: then,

the amount of P in 1 year $= (1+r)P$:

the amount of P in 2 years $=$ the amount of $(1+r)P$ in 1 year

$$= (1+r)(1+r)P = (1+r)^2P:$$

the amount of P in 3 years $=$ the amount of $(1+r)^2P$ in 1 year

$$= (1+r)(1+r)^2P = (1+r)^3P: \text{ \&c.}$$

and, the amount of P in n years $= (1+r)^nP$:

that is, $M = (1+r)^nP$, the amount:

also, $I = M - P = \{(1+r)^n - 1\}P$, the interest.

If R be assumed to represent $1+r$, or £1. together with its interest for a year, we shall have

$$M = PR^n, \text{ and } I = (R^n - 1)P,$$

which are the formulæ commonly used.

283. Cor. By means of these equations, any one of the quantities employed may be expressed in terms of the rest: and the most important are

$$P = \frac{M}{R^n}, \text{ and } n = \frac{\log M - \log P}{\log R}.$$

Ex. 1. Required the amount and compound interest, of £160, in 4 years, at 6 per cent. per annum.

Here, $P = 160$, $R = 1.06$, and $n = 4$: whence we have

$$M = 160\,(1.06)^4 = £201.\ 19s.\ 11d.\ 1152, \text{ nearly}:$$

$$I = £201.\ 19s.\ 11d.\ 1152 - £160. = £41.\ 19s.\ 11d.\ 1152, \text{ nearly}.$$

Ex. 2. If P£ at interest amount to M£ in t years: what sum must be paid down to receive P£, at the end of t years?

Let x denote the required sum: then, we shall have

$$M = PR^t, \text{ and } P = xR^t:$$

$$\text{whence, } \frac{M}{P} = \frac{PR^t}{xR^t} = \frac{P}{x}, \text{ and } x = \frac{P^2}{M},$$

a result agreeing with that of example (3) to article (276).

Ex. 3. If P£ at compound interest, rate r, double itself in n years, and at rate $2r$, in m years: find the relation between m and n.

Here, we have $2P = P(1+r)^n$, and $2 = (1+r)^n$:

also, $2P = P(1+2r)^m$, and $2 = (1+2r)^m$:

whence, $(1+2r)^m = (1+r)^n$, and $m \log(1+2r) = n \log(1+r)$:

$$\therefore \frac{m}{n} = \frac{\log(1+r)}{\log(1+2r)} > \frac{\log(1+r)}{\log(1+2r+r^2)} > \frac{\log(1+r)}{\log(1+r)^2} > \frac{1}{2}.$$

Ex. 4. A sum of money s£ is left among A, B, C, in such a manner that at the end of a, b, c years, when they respectively come of age, they are to possess equal sums; required the share of each.

Let x, y, z denote the three shares: then, we shall have

$$x + y + z = s:$$

also, $xR^a = yR^b = zR^c$, are the equations of *condition*:

whence, $y = R^{a-b}x$, and $z = R^{a-c}x$: so that,

$$x + R^{a-b}x + R^{a-c}x = s, \text{ and } \therefore\ x = \frac{s}{1 + R^{a-b} + R^{a-c}}:$$

similarly, $y = \dfrac{s}{1 + R^{b-a} + R^{b-c}}$, and $z = \dfrac{s}{1 + R^{c-a} + R^{c-b}}$.

284. When compound interest is allowed, the ***Present Worth*** will be obtained from the equation $P = \dfrac{M}{R^n}$, and the ***Discount*** from $D = M - P = M - \dfrac{M}{R^n} = \dfrac{M(R^n - 1)}{R^n}$: and when several sums are due at different periods, the *equated* time x may be found from the equation,

$$\frac{S}{R^T} + \frac{s}{R^t} + \&c. = \frac{S + s + \&c.}{R^x}:$$

as will appear from the reasoning employed in article (280).

285. We have hitherto supposed that the interest accruing from the use of money, has been converted into principal, and begun itself to bear interest, at the end of *a year:* but it is evident that the principles above explained will enable us to find the amount in a given time, at whatever equidistant periods this may be supposed to take place: thus, if t denote any number of years, and the conversion of interest into principal occur at the end of every m^{th} part of a year: we shall have $1 + \frac{r}{m}$ = the amount of £1 in that time: and therefore

$$M = \left(1 + \frac{r}{m}\right)^{mt} P = PR^{mt}.$$

If $m = 2$, or the interest be payable *half yearly*,

$$M = \left(1 + \frac{r}{2}\right)^{2t} P:$$

if $m = 4$, or the interest be payable *quarterly*,

$$M = \left(1 + \frac{r}{4}\right)^{4t} P: \text{ \&c.}$$

If $m = \infty$, or the interest be payable *every instant*,

$$M = \left(1 + \frac{r}{\infty}\right)^{\infty t} P,$$

which is merely a symbolical result: but its value is nevertheless assignable, as will now be shewn: for, by the binomial theorem, we have

$$\left(1 + \frac{r}{m}\right)^{mt} = 1 + mt\left(\frac{r}{m}\right) + \frac{mt(mt-1)}{1 \,.\, 2}\left(\frac{r}{m}\right)^2$$

$$+ \frac{mt(mt-1)(mt-2)}{1 \,.\, 2 \,.\, 3}\left(\frac{r}{m}\right)^3 + \text{\&c.}$$

$$= 1 + rt + \frac{r^2 t^2}{1 \,.\, 2} + \frac{r^3 t^3}{1 \,.\, 2 \,.\, 3} + \text{\&c.}$$

(when m is indefinitely great, and therefore 1, 2, 3, &c. may be neglected with respect to mt)

$= e^{rt}$, where $e = 2.71828$ &c., as will be demonstrated in the first Appendix.

These expressions will enable us to determine the advantages obtained, by having the money paid at short intervals of time, instead of that of a year: and without *special agreements* to the contrary, it seems *equitable* that money should bear interest from the *moment* it becomes due, and consequently that $M = Pe^{rt}$ is *theoretically* correct, although it would be attended with many inconveniences in *practice.*

286. To find in what times, a sum of money will *double* and *treble* itself, at 5 per cent., compound interest.

Here, $2P = PR^{t_2}$, and $3P = PR^{t_3}$:

$$\therefore\ 2 = R^{t_2},\ \text{and}\ 3 = R^{t_3}:\ \text{whence, we have}$$

$$t_2 = \frac{\log 2}{\log 1.05} = \frac{.3010300}{.0211893} = 14.2 \text{ years, nearly:}$$

$$t_3 = \frac{\log 3}{\log 1.05} = \frac{.4771213}{.0211893} = 22.5 \text{ years, nearly.}$$

287. Cor. Neither of the results obtained in the last article, though conveying useful and instructive information, is founded upon correct grounds consistent with the view of compound interest, inasmuch as the interval, at which interest is converted into principal, has been supposed to be previously defined; and, indeed, whenever compound interest for a fractional part of such interval is required, it is merely a matter of convention to allow such interest at all, unless the principle of simple interest be recognised at the same time: thus, if compound interest be required for $\left(t + \frac{1}{m}\right)$ years, we find the amount at the end of t years to be PR^t: and the amount of this for $\frac{1}{m}$th part of a year will be

$$= PR^t + \frac{1}{m}^{\text{th}} \text{ part of the interest of } PR^t \text{ for a year}$$

$$= PR^t + \frac{r}{m} PR^t, \text{ by the formula for simple interest,}$$

$$= \left(1 + \frac{r}{m}\right) PR^t :$$

that is, the amount of P in $\left(t + \frac{1}{m}\right)$ years, is expressed by

$$P (1 + r)^t \left(1 + \frac{r}{m}\right) :$$

and not by

$$PR^{t+\frac{1}{m}} = P (1 + r)^t (1 + r)^{\frac{1}{m}} = P (1 + r)^t \left\{1 + \frac{r}{m} + \&\text{c.}\right\},$$

according to the general expression.

288. Def. The term *Annuity* is understood to signify any *Interest of Money*, *Rent* or *Pension*, payable from time to time, at particular periods: and these payments may take place *yearly*, *half-yearly*, *quarterly*, &c.

289. *To find the amount of an annuity, left unpaid any number of years, at simple interest.*

Let A denote the annuity or annual payment: then, the amount of the first payment, which is foreborne to be received for $(n - 1)$ years, will be

$$A \left\{1 + (n - 1) r\right\} :$$

that of the second, which is foreborne $(n - 2)$ years, will be

$$A \left\{1 + (n - 2) r\right\} : \ \&\text{c.}$$

whence, the whole amount due at the end of n years, will be the sum of n terms of the arithmetical progression,

$$A \left\{1 + (n - 1) r\right\}, \quad A \left\{1 + (n - 2) r\right\}, \quad \&\text{c.}$$

$$= nA + Ar \left\{1 + 2 + 3 + \&\text{c.} (n - 1)\right\}$$

$$= nA + \frac{n (n - 1)}{1 . 2} rA : \text{ or, } M = \left\{n + \frac{n (n - 1)}{1 . 2} r\right\} A.$$

290. Def. The *Present Value* of an annuity, is that sum which, being *improved* at interest for the given time, becomes equal to the amount of the annuity.

291. *To find the present value of an annuity, to continue a given number of years, at simple interest.*

Let P denote the present value, whose amount in n years will be $(1 + nr)\,P$, by article (274): whence, according to the definition, we must have

$$(1 + nr)\,P = \left\{n + \tfrac{1}{2}n\,(n-1)\,r\right\} A:$$

and therefore $P = \dfrac{n + \frac{1}{2}n\,(n-1)\,r}{1 + nr} A.$

From each of the equations just investigated,

$$(1) \quad M = \left\{n + \tfrac{1}{2}n\,(n-1)\,r\right\} A,$$

$$(2) \quad P = \frac{n + \frac{1}{2}n\,(n-1)\,r}{1 + nr} A:$$

any one of the quantities involved, may easily be expressed in terms of the rest.

292. Cor. Since, from the equation (2), we have

$$(1 + nr)\,P = \tfrac{1}{2}nA\left\{2 + (n-1)\,r\right\}$$

$$= \tfrac{1}{2}nA\,(1 + nr) + \tfrac{1}{2}nA\,(1 - r):$$

$$\therefore\ P = \tfrac{1}{2}nA + \tfrac{1}{2}nA\left(\frac{1-r}{1+nr}\right) = \tfrac{1}{2}nA + \tfrac{1}{2}A\left(\frac{1-r}{\frac{1}{n} + r}\right):$$

and when n is infinite, or the annuity is considered to be a *Perpetuity*, we shall have

$$P = \tfrac{1}{2}nA = \infty:$$

that is, in order to purchase a *Freehold Estate* or annuity to continue for ever, it will be requisite to pay down an *infinite* sum to secure a *finite* annual payment: and consequently the view of the subject above taken cannot be correct, and recourse must be had to other principles.

293. *To find the amount of an annuity, left unpaid any number of years, at compound interest.*

Let A be the annuity: then the amount of the first payment, which is foreborne for $(n - 1)$ years, will be AR^{n-1}: of the second for $(n - 2)$ years, it will be AR^{n-2}: &c.

$\therefore$ the whole amount $= A\{1 + R + R^2 + \text{\&c. to } n \text{ terms}\}$:

$$\text{or, } M = \frac{A(R^n - 1)}{R - 1}.$$

294. *To find the present value of an annuity, to be paid for a given number of years, at compound interest.*

If P be the present value, whose amount in n years will be PR^n: then, we must evidently have

$$PR^n = \frac{A(R^n - 1)}{R - 1}, \text{ and } \therefore P = \frac{A(R^n - 1)}{R^n(R - 1)}.$$

From these two equations:

$$(1) \quad M = \frac{A(R^n - 1)}{R - 1}, \qquad (2) \quad P = \frac{A(R^n - 1)}{R^n(R - 1)}:$$

which comprise the whole theory of annuities at compound interest, any one of the quantities will become known, when the rest are given.

295. Cor. If in the expression of the last article,

$$P = \frac{A(R^n - 1)}{R^n(R - 1)} = \frac{1 - \frac{1}{R^n}}{R - 1} A,$$

we suppose n to be indefinitely great, we shall have

$$\frac{1}{R^n} = 0, \text{ and } P = \frac{A}{R - 1}:$$

which is therefore the present value or worth, of the annuity A to continue payable *for ever*.

This is the formula by which the sales or purchases of *Freehold Estates* are regulated: and it is evident that the sum

of money paid, must be *greater* or *less*, according as the rate of interest of money is *lower* or *higher*, as other considerations would suggest.

All annuities which are termed *Annuities Certain*, or in *Possession*, are treated according to the formulæ of these articles.

296. The present value of an annuity which is to commence at the end of p years, and then to be continued for q years, will manifestly be equal to its present value for $p + q$ years, diminished by its present value for p years: that is,

$$P = \frac{A}{R-1}\left\{\left(1 - \frac{1}{R^{p+q}}\right) - \left(1 - \frac{1}{R^p}\right)\right\} = \frac{A}{R-1}\left(\frac{1}{R^p} - \frac{1}{R^{p+q}}\right)$$

$$= \frac{A}{R^p(R-1)}\left(1 - \frac{1}{R^q}\right):$$

and if q be indefinitely great, or the annuity be payable *for ever*, after p years have expired, we shall have

$$P = \frac{A}{R^p(R-1)}.$$

These formulæ enable us to compute the values of *Reversions*, or of *Annuities in Reversion;* and the latter determines the value of the *Absolute Reversion* of an annuity, or of the *Fee Simple* of a freehold estate, which is to fall in at the expiration of p years.

There are also *Contingent Annuities* and *Reversions*, which depend upon some contingency, as the continuance or failure of the life of a particular individual: and if the value of the life be given, and be denoted by p, the same formulæ will give their present values.

The doctrine of *Life Annuities*, of which the case above is a very simple instance, will be briefly noticed in the first Appendix.

EXAMPLES.

(1) The amount of the sum A in n years is m, and the amount of an annuity A in the same time, is M: what sum must be paid down, to receive A yearly for n years?

At *simple interest*, we have by the preceding articles:

$$m = (1 + nr) A: \ M = \{n + \tfrac{1}{2} n(n-1) r\} A:$$

whence, if x be the sum required, we shall have

$$x = \frac{\{n + \frac{1}{2} n(n-1) r\} A}{1 + nr} = \frac{AM}{m}.$$

At *compound interest*, we have as before,

$$m = (1+r)^n A: \quad M = \frac{(1+r)^n - 1}{r} A:$$

$$\text{and } \therefore \ x = \frac{(1+r)^n - 1}{r(1+r)^n} A = \frac{AM}{m}, \text{ the same as above.}$$

(2) The present value of an annuity of 1£, to continue x years, is 10£: and the present value of an annuity of 1£, to continue $2x$ years, is 16£: required the rate of interest.

$$\text{Here, } 10 = \frac{(1+r)^x - 1}{r(1+r)^x}, \text{ and } 16 = \frac{(1+r)^{2x} - 1}{r(1+r)^{2x}}:$$

$$\therefore \frac{16}{10} = \frac{(1+r)^{2x} - 1}{r(1+r)^{2x}} \times \frac{r(1+r)^x}{(1+r)^x - 1}$$

$$= \frac{(1+r)^x + 1}{(1+r)^x}:$$

whence, $16(1+r)^x = 10(1+r)^x + 10$, and $(1+r)^x = \frac{5}{3}$:

and therefore by substitution in the first equation, we find $100r = 4$, the rate per cent.

(3) If a person, about to purchase the lease of an estate, be able continually to invest money at 4 per cent., receiving the interest half yearly: shew that if the tenant pay his rent half

yearly, the value of the lease to the purchaser is nearly 1.01 times what it would be, if the tenant paid his rent yearly.

Let P_1 and P_2 represent the values of the lease, when the rent is paid yearly, and half yearly respectively: then, we shall have

$$(1+\tfrac{1}{2}r)^{2n}P_1 = \frac{(1+r)^n-1}{r}2A:$$

$$(1+\tfrac{1}{2}r)^{2n}P_2 = \frac{(1+\tfrac{1}{2}r)^{2n}-1}{r}2A:$$

$$\text{whence, } \frac{P_2}{P_1} = \frac{(1+\tfrac{1}{2}r)^{2n}-1}{(1+r)^n-1} = 1+\tfrac{1}{4}r, \text{ nearly,}$$

$$= 1 + .01, \text{ nearly, } = 1.01, \text{ nearly:}$$

and therefore $P_2 = 1.01\,P_1$, nearly.

(4) An annuity is to commence at the end of p years, and to continue q years: find the equivalent annuity, which is to commence immediately, and to continue q years.

Let x denote the required annuity: then,

$$\text{the present value of } A = \frac{(1+r)^q-1}{r(1+r)^{p+q}}A:$$

$$\text{.......................} \quad x = \frac{(1+r)^q-1}{r(1+r)^q}x:$$

$$\text{whence, } \frac{(1+r)^q-1}{r(1+r)^q}x = \frac{(1+r)^q-1}{r(1+r)^{p+q}}A, \text{ by the question:}$$

$$\text{and } \therefore x = \frac{A}{(1+r)^p}, \text{ which is independent of } q.$$

If $p = 0$, we have $x = A$, as it manifestly ought to be.

(5) Four persons H, I, K, L contribute equal sums towards the purchase of a freehold estate: find the times that H, I, K may successively enjoy it, in order that L may be entitled to the absolute reversion.

Let x, y, z denote the required times in order: then,

$$\text{the present value of } H\text{'s interest} = \frac{A}{r}\left(1 - \frac{1}{(1+r)^z}\right):$$

$$\text{.................... } I\text{'s ....} = \frac{A}{r}\left(\frac{1}{(1+r)^z} - \frac{1}{(1+r)^{z+y}}\right):$$

$$\text{.................... } K\text{'s ....} = \frac{A}{r}\left(\frac{1}{(1+r)^{z+y}} - \frac{1}{(1+r)^{z+y+x}}\right):$$

$$\text{.................... } L\text{'s ....} = \frac{A}{r}\left(\frac{1}{(1+r)^{z+y+x}}\right):$$

and these being equal by the question, we have

$$(1+r)^z = 2, \text{ from (3) and (4), and } z = \frac{\log 2}{\log(1+r)}:$$

$$(1+r)^y = \frac{3}{2}, \text{ from (2) and (3), and } y = \frac{\log 3 - \log 2}{\log(1+r)}:$$

$$(1+r)^x = \frac{4}{3}, \text{ from (1) and (2), and } x = \frac{\log 4 - \log 3}{\log(1+r)}.$$

The same method of solution will be applicable, whatever number of persons a transaction of this kind comprises: and also, when their payments are in any assigned ratios.

For further *Applications* of Logarithms, the student is referred to the fourth chapter, and for their *Calculation*, to the seventh chapter of the *third edition* of the author's *Trigonometry:* and some additional information on *Interest* and *Annuities* may be found in the first Appendix of this work.

CHAPTER XII.

INDETERMINATE EQUATIONS AND PROBLEMS.

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297. DEF. IT has been seen in the preceding pages, that the values of the unknown symbols in equations cannot be assigned, when their number exceeds that of the equations which express their relations to each other: in such cases the equations, as well as the problems of which they are algebraical representations, become *indeterminate*, and their solutions can only be effected by introducing additional conditions at pleasure, or such as may be necessary to restrict them to the circumstances of the questions under consideration.

We will here confine our attention to a very limited view of the subject, and exhibit, by means of a few equations and problems, a mere outline of the principles which it involves.

298. DEF. An indeterminate equation of two unknown quantities of the *first degree*, is of the form

$$ax + by = c:$$

an indeterminate equation of two unknown quantities of the *second degree*, is of the form

$$ax^2 + bxy + cy^2 + dx + ey = f:$$

and similarly, of more unknown quantities and higher degrees.

Ex. 1. Given $7x + 19y = 92$, to find simultaneous integral values of x and y.

Here, $7x = 92 - 19y$, and $x = 13 - 2y - \dfrac{5y - 1}{7}$:

$\therefore$ if x be a whole number, we must have $\frac{5y-1}{7}$ a whole number:

$$\text{let } \frac{5y-1}{7} = p, \quad \therefore \; 5y = 7p + 1, \text{ and } y = p + \frac{2p+1}{5}:$$

and if y be a whole number, we must have $\frac{2p+1}{5}$ a whole number:

$$\text{let } \frac{2p+1}{5} = q, \quad \therefore \; 2p = 5q - 1, \text{ and } p = 2q + \frac{q-1}{2}:$$

and if p be a whole number, we must have $\frac{q-1}{2}$ a whole number:

$$\text{let } \frac{q-1}{2} = r, \quad \therefore \; q = 2r + 1,$$

which will evidently be a whole number, if r be so:

whence, $p = 2q + r = 5r + 2$, a whole number:

$y = p + q = 7r + 3$, a whole number:

$x = 13 - 2y - p = 5 - 19r$, a whole number:

that is, the general solution of the equation in whole numbers is

$$x = 5 - 19r, \text{ and } y = 7r + 3,$$

where the value of r may be 0, or any whole number whatever: thus,

if $r = 0$, $x = 5$, and $y = 3$:

if $r = 1$, $x = -14$, and $y = 10$:

if $r = 2$, $x = -33$, and $y = 17$:

if $r = 3$, $x = -52$, and $y = 24$: &c.

from which it appears, that the only solution in positive whole numbers is, when $x = 5$ and $y = 3$: but that, if negative values of x or y be admitted, the number of solutions is indefinite.

Ex. 2. Solve the equation $11x - 18y = 63$, in whole numbers.

Here, $11x = 63 + 18y$, $\therefore\ x = 5 + y + \dfrac{7y+8}{11}$:

let $\dfrac{7y+8}{11} = p$, $\therefore\ 7y = 11p - 8$, and $y = p - 1 + \dfrac{4p-1}{7}$:

let $\dfrac{4p-1}{7} = q$, $\therefore\ 4p = 7q + 1$, and $p = q + \dfrac{3q+1}{4}$:

let $\dfrac{3q+1}{4} = r$, $\therefore\ 3q = 4r - 1$, and $q = r + \dfrac{r-1}{3}$:

let $\dfrac{r-1}{3} = s$, $\therefore\ r = 3s + 1$:

whence, we have $r = 3s + 1$:

$$q = r + s = 4s + 1:$$

$$p = q + r = 7s + 2:$$

$$y = p - 1 + q = 11s + 2:$$

$$x = 5 + y + p = 18s + 9:$$

which will all be whole numbers, if $s = 0$, or any whole number whatever: and the general solution in whole numbers is

$$x = 18s + 9,\quad y = 11s + 2.$$

If $s = 0$, we have $x = 9$, $y = 2$:

$s = 1$, ... $x = 27$, $y = 13$:

$s = 2$, ... $x = 45$, $y = 24$:

$s = 3$, ... $x = 63$, $y = 35$: &c.

$s = -1$, ... $x = -9$, $y = -9$:

$s = -2$, ... $x = -27$, $y = -20$:

$s = -3$, ... $x = -45$, $y = -31$: &c.

from which it is manifest, that the equation admits of an infinite number of solutions in both positive and negative whole numbers: and the least positive values of x and y, which answer the purpose, are 9 and 2.

The method of solution, adopted in these two examples, is manifestly applicable to every equation of the form,

$$ax + by = c,$$

wherein a, b, c are either positive or negative whole numbers.

299. The equation $ax \pm by = c$, cannot be solved in whole numbers, unless a and b are prime to each other.

For, if possible, let $a = md$ and $b = nd$: then,

$$mdx \pm ndy = c, \text{ and } mx \pm ny = \frac{c}{d},$$

which is absurd, if the proposed equation be in its lowest terms, as is always supposed to be the case.

300. If one solution of the equation $ax + by = c$, be given, all the others may be derived from it.

For, let α, β be simultaneous values of x and y, so that

$$a\alpha + b\beta = c:$$

$$\therefore\ a(x - \alpha) + b(y - \beta) = 0, \text{ and } x - \alpha = -\frac{b}{a}(y - \beta):$$

whence, $y - \beta$ must be a multiple of a, as $\pm ra$:

$$\therefore\ x - \alpha = \mp rb, \text{ or } x = \alpha \mp rb:$$

$$y - \beta = \pm ra, \text{ or } y = \beta \pm ra:$$

from which all the values of x and y will be obtained, by giving all possible integral values to r.

301. When a and b are prime to each other, each term of the series b, $2b$, $3b$, &c. $(a - 1)b$, when divided by a, will leave a different remainder.

For, if possible let αb, βb leave the same remainder r, so that

$$\alpha b = ma + r, \text{ and } \beta b = na + r:$$

$$\therefore\ (\alpha - \beta)b = (m - n)a, \text{ and } \frac{\alpha - \beta}{m - n} = \frac{a}{b}:$$

whence, $\alpha - \beta$ is a multiple of a: which is absurd, since α, β are both less than a.

Hence these remainders must comprise all numbers from 1 to $a-1$ inclusive: and if y be less than a, a value of y can always be found such that $\frac{by-1}{a}=x$, a whole number: therefore the equation,

$$by-1=ax, \text{ or } ax-by=-1 \text{ is always possible:}$$

similarly, the equation,

$$\frac{by-(a-1)}{a}=x, \text{ or } by-a+1=ax,$$

or $a(x+1)-by=1$, and $\therefore\ ax-by=1$, is always possible: that is, when a and b are prime to each other, it is always possible to find such integral values p and q of x and y, that

$$ap-bq=\pm 1.$$

302. To find the limits of the number of solutions of $ax+by=c$, in positive whole numbers.

Let t be the number of solutions required: and let p, q be such that

$$ap-bq=1:$$

$$\text{then, } acp-bcq=c, \text{ and } abt-abt=0:$$

whence, by subtraction, we obtain

$$a(cp-bt)+b(at-cq)=c:$$

$$\therefore\ x=cp-bt, \text{ and } y=at-cq:$$

and in order that x and y may be positive integers, it is evident that bt is less than cp, and at greater than cq:

$$\text{whence, } t \text{ is less than } \frac{cp}{b}, \text{ and greater than } \frac{cq}{a}:$$

that is, the number of solutions will be expressed by the greatest integer contained in

$$\frac{cp}{b}-\frac{cq}{a}:$$

or, by the difference of the *integral parts* of these quantities, because t is less than one of them, and greater than the other: except in the case when $\frac{cp}{b}$ is a whole number, and therefore

the number of solutions is less than this difference by 1, since t is *less* than $\frac{cp}{b}$.

If p and q be such that we have $ap - bq = -1$: then will

$$-acp + bcq = c, \text{ and } abt - abt = 0:$$

$$\text{whence, } a(bt - cp) + b(cq - at) = c:$$

$$\therefore\ x = bt - cp, \text{ and } y = cq - at:$$

which prove that t is greater than $\frac{cp}{b}$, and less than $\frac{cq}{a}$: and the limits are the same as before, in an inverted order.

303. Cor. When the proposed equation is $ax - by = c$, and $ap - bq = 1$, we have

$$acp - bcq = c, \text{ and } abt - abt = 0:$$

$$\text{whence, } a(cp + bt) - b(cq + at) = c:$$

$$\therefore\ x = cp + bt, \text{ and } y = cq + at:$$

or the value of t is *unlimited* for positive values of x and y.

If only negative values of x and y be admitted, the number of solutions will be greater than $\frac{cp}{b}$ or $\frac{cq}{a}$.

304. If a and b be prime to each other, the equation

$$ax - by = \pm c,$$

is always possible: and an indefinite number of integral values may be assigned to x and y, which satisfy the equation.

For, $ax - by = \pm 1$, is always possible, by article (301):

$$\therefore\ acx - bcy = \pm c, \text{ or } ax' - by' = \pm c,$$

is always possible: also, substituting $x \pm mb$ for x', and $y \pm ma$ for y', we have

$$a(x \pm mb) - b(y \pm ma) = \pm c:$$

which is therefore possible for every integral value of m.

Hence, by means of the double sign $\pm$, and the indeterminate number m, an indefinite number of values of x and y may be found, which satisfy the equation $ax - by = \pm c$.

305. On the same supposition, the equation

$$ax + by = c,$$

is always possible in positive whole numbers, provided c be greater than $ab - a - b$.

For, if $c = ab - a - b + r$, the equation becomes

$$ax + by = ab - a - b + r,$$

which gives

$$x = b - 1 - \frac{b(y+1) - r}{a}:$$

and $\dfrac{b(y+1) - r}{a}$ may always be made equal to the integral quantity z, since

$$b(y+1) - r = az, \quad \text{or } az - b(y+1) = -r,$$

is always possible by the preceding article: but, in the equation,

$$z = \frac{b(y+1) - r}{a},$$

$y + 1$ is necessarily less than a, and therefore z is necessarily less than b, if r be a positive quantity: that is,

$$x = b - 1 - z$$

will not be negative, so long as r has a positive value, and therefore whilst c is greater than $ab - a - b$.

These two articles are of great practical utility, inasmuch as they enable us to decide at once, whether the data of a problem, dependent upon these principles, are consistent or not, as will be seen in the following pages.

306. When three or more unknown quantities are involved in one equation, as in

$$ax + by + cz = d,$$

we have $ax + by = d - cz$, the latter member of which being assumed integral, the corresponding values of x and y may be obtained by means of the preceding articles.

307. Find a number which, when divided by 2 and 3, leaves the remainders 1 and 2.

Let x denote the number required: then, we must have

$$\frac{x-1}{2} \text{ and } \frac{x-2}{3} \text{ equal to whole numbers:}$$

$$\text{whence, assuming } \frac{x-1}{2} = p, \text{ or } x = 2p + 1,$$

$$\text{we must have } \frac{x-2}{3} = \frac{2p-1}{3} = \text{a whole number:}$$

$$\text{let } \frac{2p-1}{3} = q, \therefore 2p = 3q + 1, \text{ and } p = q + \frac{q+1}{2}:$$

$$\text{let } \frac{q+1}{2} = r, \quad \therefore q = 2r - 1:$$

$$\text{whence } p = q + r = 3r - 1:$$

$$\text{and } x = 2p + 1 = 6r - 1:$$

or, the required number is always of the form $6r - 1$, the value of r being assigned at pleasure. If $r = 1$, the number required $= 5$, which is the least positive whole number that satisfies the conditions.

308. In how many ways can a person pay a bill of £12, with crowns and guineas?

Let x and y denote the numbers of crowns and guineas:

$$\text{then, } 5x + 21y = 240, \text{ by the question:}$$

$$\therefore x = 48 - 4y - \frac{y}{5} = \text{a whole number:}$$

$$\text{let } \frac{y}{5} = p, \quad \therefore y = 5p, \text{ and } x = 48 - 21p:$$

and these forms comprise all the possible solutions:

$$\text{if } p = 0, \quad x = 48 \quad \text{and} \quad y = 0:$$
$$p = 1, \quad x = 27 \quad \text{and} \quad y = 5:$$
$$p = 2, \quad x = 6 \quad \text{and} \quad y = 10.$$

If higher values be given to p, the values of x become negative, which are admissible only on the supposition, that the person may receive back crowns, whilst he pays guineas: and thus it is manifestly possible to pay the bill in an infinite number of ways.

309. If I have nine half guineas and six half crowns in my purse, how may I pay a debt of £4. 11*s.* 6*d.*?

Let x = the number of half guineas, y = that of half crowns:

then, $21x + 5y = 183$, expressed in sixpences:

$\therefore\ 5y = 183 - 21x$, and $y = 36 - 4x - \dfrac{x-3}{5} =$ a whole number:

let $\dfrac{x-3}{5} = p$, $\therefore\ x = 5p + 3$, and $y = 24 - 21p$:

$$\text{if } p = 0, \quad x = 3 \quad \text{and} \quad y = 24,$$

which are excluded by the restrictions of the question:

$$\text{if } p = 1, \quad x = 8 \quad \text{and} \quad y = 3,$$

which are within the prescribed limits.

310. Find two fractions, whose denominators shall be 7 and 9, and their sum equal to $\dfrac{19}{21}$.

Let x and y denote the numerators of the required fractions: then, we have

$$\frac{x}{7} + \frac{y}{9} = \frac{19}{21}, \text{ or } 9x + 7y = 57:$$

whence, $y = 8 - x - \dfrac{2x-1}{7} =$ a whole number:

$$\text{let } \frac{2x-1}{7} = p, \ \therefore\ 2x = 7p + 1, \ \text{and } x = 3p + \frac{p+1}{2}:$$

$$\text{let } \frac{p+1}{2} = q, \ \therefore\ p = 2q - 1:$$

$$\therefore\ x = 3p + q = 7q - 3, \ \text{and } y = 8 - x - p = 12 - 9q.$$

If $q = 1$, we have $x = 4$ and $y = 3$, and these are the only positive values of x and y which satisfy the equation.

If $q = 2$, we have $x = 11$, $y = -6$:
$q = 3$, ... $x = 18$, $y = -15$:
$q = 4$, ... $x = 25$, $y = -24$: &c.

that is, if the *difference* of the fractions be $\frac{19}{21}$, the number of solutions is unlimited.

311. The sum of two numbers is 78; one of them is divisible by 5, and the other by 3: how many pairs of numbers satisfy these conditions?

Let x = one of the numbers, then $78 - x$ = the other: and we must have both

$$\frac{x}{5} \text{ and } \frac{78-x}{3} \text{ equal to whole numbers:}$$

$$\text{assume } \frac{x}{5} = p, \ \therefore\ x = 5p: \ \text{and } \frac{78-5p}{3} = 26 - p - \frac{2p}{3}$$

must be a whole number:

$$\text{let } \frac{2p}{3} = q, \ \therefore\ 2p = 3q, \ \text{and } p = q + \frac{q}{2}:$$

$$\text{let } \frac{q}{2} = r, \ \therefore\ q = 2r, \ \text{and } p = 3r:$$

whence $x = 15r$, and $78 - x = 78 - 15r$, express generally the numbers required.

$$\text{If } r = 1, \text{ we have } x = 15, \quad 78 - x = 63:$$
$$r = 2, \ \ldots\ldots\ldots \ x = 30, \quad 78 - x = 48:$$
$$r = 3, \ \ldots\ldots\ldots \ x = 45, \quad 78 - x = 33:$$
$$r = 4, \ \ldots\ldots\ldots \ x = 60, \quad 78 - x = 18:$$
$$r = 5, \ \ldots\ldots\ldots \ x = 75, \quad 78 - x = 3:$$

that is, there are *five* pairs of numbers answering the condition, when the negative values, corresponding to higher values of r, are excluded.

312. A wheel in 36 revolutions, passes over 29 yards: and in x of these revolutions, it describes z yds. + y ft. + 5 in.: find the values of x, y, z.

Since 36 revolutions pass over 87 feet, we shall have

$$3z + y + \frac{5}{12} = \frac{87}{36}x = \frac{29}{12}x:$$

$$\text{or, } 36z + 12y + 5 = 29x:$$

whence, $12y = 29x - 36z - 5$, and $y = 2x - 3z + \dfrac{5x - 5}{12}$:

$$\text{assume } \frac{5x-5}{12} = 5p, \text{ and } \therefore \ x = 12p + 1:$$

$$\therefore \ y = 29p + 2 - 3z:$$

wherein the values of p and z may be taken at pleasure, provided y be less than 3, or $29p + 2 - 3z$ be less than 3, and therefore z be greater than $\dfrac{29p - 1}{3}$.

$$\text{Let } p = 1, \quad z = 10, \quad \therefore \ x = 13, \quad y = 1:$$
$$p = 2, \quad z = 20, \quad \therefore \ x = 25, \quad y = 0:$$
$$p = 3, \quad z = 29, \quad \therefore \ x = 37, \quad y = 2:$$
$$p = 4, \quad z = 39, \quad \therefore \ x = 49, \quad y = 1:$$
$$p = 5, \quad z = 49, \quad \therefore \ x = 61, \quad y = 0:$$
$$p = 6, \quad z = 58, \quad \therefore \ x = 73, \quad y = 2:$$
$$p = 7, \quad z = 68, \quad \therefore \ x = 85, \quad y = 1: \ \&c.$$

whence, the solution required is $x=13$, $y=1$, and $z=10$; the second giving $y=0$, and in all the rest, the value of x is beyond its proper limit.

313. Indeterminate equations of two unknown quantities of the second degree, are comprised in the forms:

$$(1)\quad y=\frac{a}{b+cx},\quad (2)\quad y=\frac{a+bx}{c+dx},\quad (3)\quad y^2=a+bx+cx^2:$$

the first and second of which admit of obvious solutions by means of the preceding principles: and we will exemplify that of the third, in some of the following problems.

314. To find two square numbers, whose sum shall be a square number.

Let $x^2+y^2=z^2$, $\therefore\ x^2=z^2-y^2=(z+y)(z-y)$:

and $mxx=(z+y)\,m(z-y)$:

whence, assuming $mx=z+y$, and $x=m(z-y)$, we shall have

$$z+y=m^2(z-y):$$

$$\therefore\ (m^2+1)y=(m^2-1)z=(m^2-1)(mx-y)$$

$$=(m^2-1)\,mx-(m^2-1)y:$$

$$\therefore\ 2m^2y=(m^2-1)mx,\ \text{and}\ x=\frac{2my}{m^2-1}:$$

to obtain whole numbers, let $y=m^2-1$, and we have $x=2m$: that is, the general forms of the two numbers will be

$$x=2m,\ \text{and}\ y=m^2-1.$$

If $m=1$, we have $x=2$, $y=0$, and $z=2$:

$m=2$, ... $x=4$, $y=3$, and $z=5$:

$m=3$, ... $x=6$, $y=8$, and $z=10$:

$m=4$, ... $x=8$, $y=15$, and $z=17$: &c.

If fractions be not excluded, we have only to assign any particular value to y, and then to proceed as above.

315. Cor. If the sum of the two squares be given, we have

$$x = \frac{2m}{m^2+1}z, \text{ and } y = \frac{m^2-1}{m^2+1}z.$$

Ex. Find the values of x and y, which will satisfy the equation $x^2 + y^2 = 10^2$.

Here, $x = \dfrac{2m}{m^2+1}10$, and $y = \dfrac{m^2-1}{m^2+1}10$: wherein the value of m may be assigned at pleasure.

If $m = 1$, we have $x = 10$, and $y = 0$:

$m = 2$, ... $x = 8$, and $y = 6$:

$m = 3$, ... $x = 6$, and $y = 8$:

$m = 4$, ... $x = \dfrac{80}{17}$, and $y = \dfrac{150}{17}$: &c.

316. To find two square numbers, whose difference shall be a square number.

Let $x^2 - y^2 = z^2$, and $\therefore (x+y)m(x-y) = mzz$:

whence, assuming $x + y = mz$, and $m(x-y) = z$,

we have $x + y = m^2(x-y)$, and $(m^2+1)y = (m^2-1)x$:

$$\therefore x = \frac{m^2+1}{m^2-1}y:$$

and if $y = m^2 - 1$, then will $x = m^2 + 1$, and $z = 2m$.

If $m = 1$, we have $x = 2$, $y = 0$, and $z = 2$:

$m = 2$, ... $x = 5$, $y = 3$, and $z = 4$:

$m = 3$, ... $x = 10$, $y = 8$, and $z = 6$:

$m = 4$, ... $x = 17$, $y = 15$, and $z = 8$: &c.

For fractional values we have only to proceed as in article (314).

317. Cor. If the difference of the two squares be given, we have

$$m(x+y)=m^2 z,$$

$$m(x-y)=z:$$

whence, $2mx=(m^2+1)z$, and $x=\dfrac{m^2+1}{2m}z$:

also, $2my=(m^2-1)z$, and $y=\dfrac{m^2-1}{2m}z$.

Ex. Determine the simultaneous values of x and y, which will satisfy the equation $x^2-y^2=24^2$.

Here, $x=\dfrac{m^2+1}{2m}24$, and $y=\dfrac{m^2-1}{2m}24$,

where the value of m may be assumed at pleasure.

If $m=1$, we have $x=24$, and $y=0$:

$m=2$, ... $x=30$, and $y=18$:

$m=3$, ... $x=40$, and $y=32$:

$m=4$, ... $x=51$, and $y=45$: &c.

These articles include the formulæ for the determination of the rational values of the sides of right-angled triangles, and are simple instances of what is termed the *Diophantine Analysis:* so called from *Diophantus*, a celebrated mathematician of *Alexandria*, who is supposed to have flourished about the year 280 of the Christian æra.

318. The difference between the squares of the ages of two persons at one period was 45: and at another it was 159: required the age of each.

Here we have

$$x^2-y^2=45, \text{ or } (x-y)(x+y)=45:$$

$$x'^2-y'^2=159, \text{ or } (x'-y')(x'+y')=159:$$

but since $x'-y'=x-y$, and 45 and 159 have the common factors 1 and 3, we must have

$x-y=1$ or 3, and therefore $x+y=45$ or 15:

whence, $x = 23$ and $y = 22$: or $x = 9$ and $y = 6$:

also, $x' - y' = 1$ or 3, and therefore $x' + y' = 159$ or 53:

whence, $x' = 80$ and $y' = 79$: or $x' = 28$ and $y' = 25$:

that is, at the first period, their ages were 9 and 6, and at the second 28 and 25: or, at the first period they were 23 and 22, and at the second 80 and 79.

If the given differences had been prime to each other, there would have been only one solution of the problem.

319. To find three square numbers in arithmetical progression.

Let x^2, y^2, z^2 denote the required squares: then,

$$x^2 + z^2 = 2y^2:$$

assume $x = p + q$ and $z = p - q$, so that $p^2 + q^2 = y^2$, which by article (314), will be satisfied in whole numbers, by making

$$p = 2m \text{ and } q = m^2 - 1, \text{ and therefore } y = m^2 + 1:$$

whence, the general solution will be expressed by

$$x = 2m + m^2 - 1, \quad z = 2m - m^2 + 1, \text{ and } y = m^2 + 1.$$

Ex. If $m = 1$, we have $x^2 = 4$, $y^2 = 4$ and $z^2 = 4$, which can scarcely be said to be in arithmetical progression:

if $m = 2$, we find $x = 7$, $y = 5$ and $z = 1$,

whose squares are 49, 25 and 1, forming an arithmetical progression:

if $m = 3$, we have $x = 14$, $y = 10$ and $z = -2$,

whose squares are 196, 100 and 4, in arithmetical progression: &c.

If the student be desirous of more information upon these subjects, he may consult *Barlow*'s Theory of Numbers, and *Leybourne*'s Mathematical Repository.

CHAPTER XIII.

CONTINUED FRACTIONS.

320. Def. Any expression written in the form,

$$p + \cfrac{1}{q + \cfrac{1}{r + \cfrac{1}{s + \&c.}}}$$

is called a *Continued Fraction*: and it is said to be *rational* or *irrational*, according as the number of terms comprised in it is *finite* or *infinite*: also,

$$p, \quad p + \frac{1}{q}, \quad p + \cfrac{1}{q + \cfrac{1}{r}}, \quad \&c.$$

when reduced to their simplest forms

$$p, \quad \frac{pq+1}{q}, \quad \frac{pqr+p+r}{qr+1}, \quad \&c.$$

are termed the corresponding *Converging Fractions*.

We shall presently see in what manner, expressions of this kind originate: and the subsequent articles will point out their use, both in approximating to the values of ratios exhibited by large numbers, and in resolving indeterminate equations of the first and second degrees: of the latter of which, the solutions in whole numb[illegible] opendix, could not be easily obtained by

321. To express a vulgar fraction, in the form of a continued fraction.

Let $\frac{a}{b}$ be the proposed fraction: then, proceeding as in article (53), we have

$$\begin{aligned} a &= pb + c: \\ b &= qc + d: \\ c &= rd + e: \\ d &= se + f: \quad \&c. \end{aligned}$$

$$\text{whence,} \quad \frac{a}{b} = p + \frac{c}{b} = p + \frac{c}{qc + d}$$

$$= p + \cfrac{1}{q + \cfrac{d}{c}}$$

$$= p + \cfrac{1}{q + \cfrac{d}{rd + e}} = p + \cfrac{1}{q + \cfrac{1}{r + \cfrac{e}{d}}}$$

$$= p + \cfrac{1}{q + \cfrac{1}{r + \cfrac{e}{se + f}}} = p + \cfrac{1}{q + \cfrac{1}{r + \cfrac{1}{s + \&c.}}}$$

Hence, the converging fractions will evidently be

$$\frac{p}{1}, \quad \frac{pq+1}{q}, \quad \frac{pqr + p + r}{qr + 1}, \quad \frac{pqrs + pq + ps + rs + 1}{qrs + q + s}, \quad \&c.$$

The integers p, q, r, &c. are called the *Quotients* or *Partial Quotients*: and each of these with its connected fraction, as $p + \frac{c}{b}$, $q + \frac{d}{c}$, $r + \frac{e}{d}$, &c. is termed a *Complete Quotient*: and it is evident that every irreducible fraction may be expressed as a continued fraction by the same method.

Ex. Represent $\frac{365}{224}$ in the form of a continued fraction, and find the converging fractions.

Here,

$$\begin{array}{l}
224)\,365\,(1 \\
\quad 224 \\
\quad 141)\,224\,(1 \\
\qquad 141 \\
\qquad 83)\,141\,(1 \\
\qquad\quad 83 \\
\qquad\quad 58)\,83\,(1 \\
\qquad\qquad 58 \\
\qquad\qquad 25)\,58\,(2 \\
\qquad\qquad\quad 50 \\
\qquad\qquad\quad 8)\,25\,(3 \\
\qquad\qquad\qquad 24 \\
\qquad\qquad\qquad 1)\,8\,(8 \\
\qquad\qquad\qquad\quad 8
\end{array}$$

$$\therefore \frac{365}{224} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{3 + \frac{1}{8}}}}}} :$$

and the converging fractions taken in order, will be

$$\frac{1}{1},\ \frac{2}{1},\ \frac{3}{2},\ \frac{5}{3},\ \frac{13}{8},\ \frac{44}{27},\ \frac{365}{224},$$

the last of which is the fraction proposed: and it will readily appear that each of these is more nearly equal to it, than that which immediately precedes.

Even in this simple instance, the process of finding the converging fractions is tedious: and we will now deduce a method by which all embarrassment may be avoided.

322. Let the continued fraction belonging to $\frac{a}{b}$, be

$$\alpha + \cfrac{1}{\beta + \cfrac{1}{\gamma + \&c.}},$$

where the quotients taken in order are

$$\alpha, \beta, \gamma, \delta, \&c., \lambda, \mu, \nu, \rho, \&c.:$$

then, by article (320), the first three converging fractions will be

$$\frac{\alpha}{1}, \quad \frac{\alpha\beta + 1}{\beta}, \quad \frac{(\alpha\beta + 1)\gamma + \alpha}{\beta\gamma + 1}:$$

in which we observe that the third fraction is deduced from the two preceding, according to a certain law: and since

$$\alpha = \cfrac{1}{0 + \cfrac{1}{\alpha}},$$

if we write down the quotients and converging fractions in the form,

$$\begin{array}{cccc} & \alpha, & \beta, & \gamma, \\ \frac{1}{0}, & \frac{\alpha}{1}, & \frac{\alpha\beta + 1}{\beta}, & \frac{(\alpha\beta + 1)\gamma + \alpha}{\beta\gamma + 1}, \end{array}$$

we see that each fraction is derived from the two preceding, by multiplying the numerator of the last by the quotient, and adding the numerator of the last but one, for a new numerator: and by multiplying the denominator of the last by the quotient, and adding the denominator of the last but one, for a new denominator: suppose this law to continue, and let

$$\frac{p_1}{q_1}, \quad \frac{p_2}{q_2}, \quad \frac{p_3}{q_3}, \quad \frac{p_4}{q_4}, \quad \&c.$$

be consecutive converging fractions, and μ, ν be the quotients corresponding to $\frac{p_3}{q_3}$ and $\frac{p_4}{q_4}$, so that

$$\frac{p_3}{q_3} = \frac{\mu p_2 + p_1}{\mu q_2 + q_1}:$$

then, since $\frac{p_4}{q_4}$ differs from $\frac{p_3}{q_3}$, only by the substitution of $\mu + \frac{1}{\nu}$ in the place of μ, we shall have

$$\frac{p_4}{q_4} = \frac{\left(\mu + \frac{1}{\nu}\right) p_2 + p_1}{\left(\mu + \frac{1}{\nu}\right) q_2 + q_1}$$

$$= \frac{(\mu\nu + 1) p_2 + \nu p_1}{(\mu\nu + 1) q_2 + \nu q_1} = \frac{\nu(\mu p_2 + p_1) + p_2}{\nu(\mu q_2 + q_1) + q_2} = \frac{\nu p_3 + p_2}{\nu q_3 + q_2},$$

which is of the same form as the value of $\frac{p_3}{q_3}$, with ν in the place of μ: whence we infer by induction, that the law as above enunciated is universally true: and we have the following general Rule.

Write down the quotients and the values of the converging fractions, in the following form:

$$\begin{array}{ccccc} & \alpha, & \beta, & \gamma, & \&c. \\ \frac{1}{0}, & \frac{\alpha}{1}, & \frac{\alpha\beta + 1}{\beta}, & \frac{(\alpha\beta + 1)\gamma + \alpha}{\beta\gamma + 1}, & \&c. \end{array}$$

$$\begin{array}{cccc} \kappa, & \lambda, & \mu, & \&c. \\ \frac{p_1}{q_1}, & \frac{p_2}{q_2}, & \frac{p_3}{q_3}, & \&c. \end{array}$$

the first quantity in the second line being introduced merely for the sake of uniformity: then, any numerator is found

from the two which precede it, by multiplying the numerator of the latter by the quotient, and adding the numerator of the former: and the same process is applied to the denominator.

323. Cor. If at any stage of the operation, we substitute the *complete quotient* z, we shall have

$$\frac{a}{b} = \frac{zp_2 + p_1}{zq_2 + q_1},$$

which is the *complete fraction*, as appears from article (321).

Ex. 1. From the fraction $\frac{365}{224}$, we have had the quotients,

$$1,\quad 1,\quad 1,\quad 1,\quad 2,\quad 3,\quad 8:$$

$$\text{and}\quad \frac{1}{0}:\quad \frac{1}{1},\quad \frac{2}{1},\quad \frac{3}{2},\quad \frac{5}{3},\quad \frac{13}{8},\quad \frac{44}{27},\quad \frac{365}{224},$$

are the converging fractions, all derived immediately by the application of the rule: the first quantity $\frac{1}{0}$ being finally rejected, as only introduced for a subsidiary purpose.

Ex. 2. Find the fractions converging to $\frac{314159}{100000}$.

Here, we have the following quotients,

$$3,\quad 7,\quad 15,\quad 1,\quad \&c.:$$

also, $\frac{1}{0}$: $\frac{3}{1}$, $\frac{22}{7}$, $\frac{333}{106}$, $\frac{355}{113}$, &c.: by the general rule:

that is, the fractions successively converging to the value of

$$\frac{314159}{100000}$$

$$\text{are}\quad \frac{3}{1},\quad \frac{22}{7},\quad \frac{333}{106},\quad \frac{355}{113},\quad \&c.:$$

and they will be found to be alternately less and greater than the true value, but more and more nearly equal to it in succession.

These are approximations to the value of the ratio of the circumference of a circle to its diameter, which is usually represented by the symbol π; the second being $\frac{22}{7}$ is that of *Archimedes*, and the fourth $\frac{355}{113}$, which will be found to be a very close approximation, is due to *Metius*: but the exact value of π being an incommensurable quantity, the entire number of converging fractions would be *infinite*, and the continued fraction *irrational*.

324. *If* $\frac{p_1}{q_1}$ *and* $\frac{p_2}{q_2}$ *be any two consecutive converging fractions: then will* $p_1 q_2 - p_2 q_1 = \pm 1$.

For, if $\frac{p_1}{q_1}$, $\frac{p_2}{q_2}$, $\frac{p_3}{q_3}$, &c. be consecutive converging fractions, and κ, λ, μ, &c. be the corresponding quotients: we have

$$p_3 = \mu p_2 + p_1, \quad \therefore \ p_3 q_2 = \mu p_2 q_2 + p_1 q_2:$$

$$q_3 = \mu q_2 + q_1, \quad \therefore \ p_2 q_3 = \mu p_2 q_2 + p_2 q_1:$$

whence, by subtraction, we obtain

$$p_3 q_2 - p_2 q_3 = -(p_2 q_1 - p_1 q_2):$$

similarly, $p_4 q_3 - p_3 q_4 = -(p_3 q_2 - p_2 q_3)$

$$= p_2 q_1 - p_1 q_2: \ \&c.:$$

that is, the difference of the product of each numerator and successive denominator, and the product of each denominator and successive numerator, is always of the same magnitude, abstracting the algebraical sign:

but, from $\frac{\alpha}{1}$ and $\frac{\alpha\beta + 1}{\beta}$, we have

$\alpha\beta - (\alpha\beta + 1) = -1$: and from $\frac{\alpha\beta + 1}{\beta}$ and $\frac{(\alpha\beta + 1)\gamma + \alpha}{\beta\gamma + 1}$, we obtain

$$(\alpha\beta + 1)(\beta\gamma + 1) - (\alpha\beta + 1)\beta\gamma - \alpha\beta = +1:$$

whence, for every two consecutive converging fractions, as $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$, we shall have

$$p_1 q_2 - p_2 q_1 = \pm 1:$$

the upper or lower sign being used, according as $\frac{p_1}{q_1}$ is found in an *even* or *odd* place, from the beginning of the series.

325. *The successive converging fractions are all in their lowest terms.*

For, if possible, let the terms of the fraction $\frac{p_1}{q_1}$, have a common measure d: then, since d measures p_1 and q_1, it will measure $p_1 q_2$ and $p_2 q_1$, and therefore their difference $p_1 q_2 - p_2 q_1$, or ± 1: which is impossible, if d be greater than 1: that is, $\frac{p_1}{q_1}$ is in its lowest terms: and similarly of the rest.

326. *The successive converging fractions are alternately less and greater than the true value of the fraction: but they become more and more nearly equal to it.*

For, let $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$ be two fractions successively converging to $\frac{a}{b}$, and let z be the complete quotient corresponding to $\frac{p_3}{q_3}$:

$$\text{then, by article (323), } \frac{a}{b} = \frac{z p_2 + p_1}{z q_2 + q_1}:$$

$$\text{whence, } \frac{a}{b} - \frac{p_1}{q_1} = \frac{z p_2 + p_1}{z q_2 + q_1} - \frac{p_1}{q_1}$$

$$= \frac{z p_2 q_1 + p_1 q_1 - z p_1 q_2 - p_1 q_1}{q_1 (z q_2 + q_1)} = \frac{z (p_2 q_1 - p_1 q_2)}{q_1 (z q_2 + q_1)}$$

$$= \frac{\mp z}{q_1 (z q_2 + q_1)}:$$

$$\text{also, } \frac{a}{b} - \frac{p_2}{q_2} = \frac{x p_2 + p_1}{x q_2 + q_1} - \frac{p_2}{q_2}$$

$$= \frac{x p_2 q_2 + p_1 q_2 - x p_2 q_2 - p_2 q_1}{q_2 (x q_2 + q_1)} = \frac{p_1 q_2 - p_2 q_1}{q_2 (x q_2 + q_1)}$$

$$= \frac{\pm 1}{q_2 (x q_2 + q_1)}:$$

from which it appears that $\frac{a}{b} - \frac{p_1}{q_1}$ and $\frac{a}{b} - \frac{p_2}{q_2}$ have different algebraical signs: and therefore the successive converging fractions are alternately less and greater than the true value of the fraction proposed.

Also, because q_2 is greater than q_1, and x is not less than 1, it follows that the numerical value of $\frac{\pm 1}{q_2 (x q_2 + q_1)}$ is less than that of $\frac{\mp x}{q_1 (x q_2 + q_1)}$: and consequently the successive converging fractions become more and more nearly equal to the proposed fraction.

327. Cor. Since $\frac{a}{b} - \frac{p_2}{q_2} = \frac{\pm 1}{q_2 (x q_2 + q_1)}$, and x cannot be less than 1, it is evident that the difference between $\frac{a}{b}$ and its approximate value $\frac{p_2}{q_2}$, can never be greater than $\frac{\pm 1}{q_2 (q_2 + q_1)}$: and therefore, *a fortiori*, it is not greater than $\frac{\pm 1}{q_1 q_2}$ nor $\frac{\pm 1}{q_2^2}$.

328. *No fraction nearer to the true value, can lie between two successive converging fractions, unless it be expressed in higher terms than either of them.*

For, if possible, let $\frac{r}{s}$ be an interpolation between $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$, where s is less than q_2:

then, $\frac{r}{s} - \frac{p_1}{q_1}$ is less than $\frac{p_2}{q_2} - \frac{p_1}{q_1}$:

$\therefore \frac{rq_1 - sp_1}{sq_1}$ is less than $\frac{p_2 q_1 - p_1 q_2}{q_1 q_2}$ and $\therefore$ less than $\frac{\pm 1}{q_1 q_2}$:

but $rq_1 - sp_1$ being an integer, is greater than ± 1:

$\therefore \frac{rq_1 - sp_1}{sq_1}$ must be greater than $\frac{\pm 1}{sq_1}$:

whence, it follows that $\frac{\pm 1}{q_1 q_2}$ is greater than $\frac{\pm 1}{sq_1}$:

or, $\frac{1}{q_2}$ is greater than $\frac{1}{s}$: and therefore s is greater than q_2, which is contrary to the hypothesis:

$\therefore \frac{r}{s}$ is not an interpolation between $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$.

329. If $\frac{p_1}{q_1}$, $\frac{p_2}{q_2}$, $\frac{p_3}{q_3}$, &c., $\frac{p_n}{q_n}$ represent all the successive fractions converging to the value of $\frac{a}{b}$; then will

$$\frac{a}{b} = \frac{p_1}{q_1} + \frac{1}{q_1 q_2} - \frac{1}{q_2 q_3} + \text{\&c.} \mp \frac{1}{bq_n}.$$

For, $\frac{p_2}{q_2} - \frac{p_1}{q_1} = \frac{1}{q_1 q_2}$:

$$\frac{p_3}{q_3} - \frac{p_2}{q_2} = -\frac{1}{q_2 q_3}:$$

$$\text{\&c.} = \text{\&c.}$$

$$\frac{a}{b} - \frac{p_n}{q_n} = \mp \frac{1}{bq_n}:$$

whence, by addition and transposition, we obtain

$$\frac{a}{b} = \frac{p_1}{q_1} + \frac{1}{q_1 q_2} - \frac{1}{q_2 q_3} + \text{\&c.} \mp \frac{1}{bq_n}.$$

330. It has appeared in the preceding articles, that

$$\frac{p_1}{q_1},\ \frac{p_2}{q_2},\ \frac{p_3}{q_3},\ \frac{p_4}{q_4},\ \&c.$$

are alternately less and greater than the true value of $\frac{a}{b}$: whence we may separate these fractions into two classes,

$$\frac{p_1}{q_1},\ \frac{p_3}{q_3},\ \frac{p_5}{q_5},\ \&c.$$

which are all less than $\frac{a}{b}$: and

$$\frac{p_2}{q_2},\ \frac{p_4}{q_4},\ \frac{p_6}{q_6},\ \&c.$$

which are all greater than $\frac{a}{b}$:

also, if α, β, γ, δ, &c. be the quotients corresponding to these convergents, we shall have

$$p_3 = \gamma p_2 + p_1,\quad q_3 = \gamma q_2 + q_1:\ \&c.$$

$$p_1 q_2 - p_2 q_1 = -1,\quad p_3 q_4 - p_4 q_3 = -1:\ \&c.$$

$$\text{whence, } \frac{p_3}{q_3} - \frac{p_1}{q_1} = \frac{p_3 q_1 - p_1 q_3}{q_1 q_3}$$

$$= \frac{(\alpha p_2 + p_1)\, q_1 - (\alpha q_2 + q_1)\, p_1}{q_1 q_3}$$

$$= \frac{\alpha\,(p_2 q_1 - p_1 q_2)}{q_1 q_3} = \frac{\gamma}{q_1 q_3}:\ \&c.:$$

and consequently when the quotients are equal to 1, it will be impossible to insert between any two consecutive terms of either of these sets of convergents, a fraction whose denominator lies between their denominators.

But if the quotients be greater than 1, as for instance, when $\gamma = 4$: we shall have $p_3 = 4p_2 + p_1$, and $q_3 = 4q_2 + q_1$:

and therefore the three fractions

$$\frac{p_2 + p_1}{q_2 + q_1}, \quad \frac{2p_2 + p_1}{2q_2 + q_1}, \quad \frac{3p_2 + p_1}{3q_2 + q_1},$$

will be all intermediate to $\frac{p_1}{q_1}$ and $\frac{p_3}{q_3}$: and having their denominators less than q_3, they will be interpolations between the two convergents $\frac{p_1}{q_1}$ and $\frac{p_3}{q_3}$.

Also, since $\frac{p_2 + p_1}{q_2 + q_1} - \frac{p_1}{q_1} = \frac{1}{q_1(q_2 + q_1)}$:

$$\frac{2p_2 + p_1}{2q_2 + q_1} - \frac{p_2 + p_1}{q_2 + q_1} = \frac{1}{(q_2 + q_1)(2q_2 + q_1)}: \text{ \&c.}$$

it is evident that all these fractions increase from $\frac{p_1}{q_1}$ towards $\frac{p_3}{q_3}$, and that no further interpolations can take place between any two that are consecutive.

Again, because $\frac{p_1}{q_1} - \frac{p_2}{q_2} = -\frac{1}{q_1 q_2}$:

$$\frac{p_2 + p_1}{q_2 + q_1} - \frac{p_2}{q_2} = -\frac{1}{q_2(q_2 + q_1)}: \text{ \&c.}$$

it is manifest that all these fractions are less than $\frac{p_2}{q_2}$: that they become in succession more and more nearly equal to it: and that no further interpolations can take place between any one of them and $\frac{p_2}{q_2}$.

Whence, upon the whole it appears, that though fractions may be found which are more nearly equal to the true value than a convergent which is *greater* or *less*, they will not be so nearly equal to it, as the succeeding convergent which is *less* or *greater*, according to the tenor of articles (326) and (328).

331. To find the value of the irrational periodic continued fraction,

$$\cfrac{1}{p+\cfrac{1}{p+\cfrac{1}{p+\&c.}}}$$

Let x = the value required: then, we have

$$x = \frac{1}{p+x}, \text{ or } x^2 + px = 1:$$

whence, $x = -\frac{1}{2}p + \frac{1}{2}\sqrt{p^2+4}$, the negative value being rejected, because x is a positive quantity.

From this example, we shall have $\frac{1}{2}\sqrt{p^2+4} = \frac{1}{2}p + x$, expressed in the form of a continued fraction:

thus, if $p = 2$, we have the square root of 2 expressed by

$$\sqrt{2} = 1 + \cfrac{1}{2+\cfrac{1}{2+\&c.}} \textit{ in infinitum.}$$

332. To find the value of the irrational periodic continued fraction,

$$\cfrac{1}{p+\cfrac{1}{q+\cfrac{1}{p+\cfrac{1}{q+\&c.}}}}$$

Here, we have $x = \cfrac{1}{p+\cfrac{1}{q+x}} = \dfrac{q+x}{pq+px+1}$:

$$\therefore\ px^2 + pqx = q, \text{ and } x = -\tfrac{1}{2}q + \frac{1}{2p}\sqrt{p^2q^2+4pq}.$$

Hence, $\dfrac{1}{2p}\sqrt{p^2q^2+4pq} = \frac{1}{2}q + x$, will convert the former member into a continued fraction: thus, if $p = 2$ and $q = 3$, we shall have

$$\sqrt{15} = 3 + 2\left\{\cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{3 + \&c.}}}}\right.$$

333. To find the value of the irrational mixed periodic continued fraction

$$\cfrac{1}{p + \cfrac{1}{q + \cfrac{1}{r + \cfrac{1}{q + \cfrac{1}{r + \&c.}}}}}$$

Here, let $x = \dfrac{1}{p+y}$:

and $y = \cfrac{1}{q + \cfrac{1}{r+y}} = \dfrac{r+y}{qr+qy+1}$:

then, the value of y being found from the latter equation and substituted in the former, we shall obtain the value of x.

334. From the equation, $x - a = \cfrac{p}{q + \cfrac{p}{q + \&c.}}$,

we obtain $x = \frac{1}{2}(2a - q + \sqrt{q^2 + 4p})$,

and $\therefore \frac{1}{2}(\sqrt{q^2+4p} - q) = x - a$:

from which, by making $q = 2a$, we have

$$\sqrt{a^2 + p} = a + \cfrac{p}{2a + \cfrac{p}{2a + \&c.}}$$

335. To express a numerical surd in the form of a continued fraction.

Let x denote the proposed surd, and let p be such a number that $x - p$ is less than 1: then, x is less than

$$p + 1 = p + \frac{1}{x'},$$

suppose, where x' is greater than 1:

let q be such a number that $x' - q$ is less than 1: then, x' is less than $q + 1 = q + \dfrac{1}{x''}$, suppose, where x'' is greater than 1:

$$\therefore\ x = p + \frac{1}{x'}$$

$$= p + \cfrac{1}{q + \cfrac{1}{x''}}:$$

let r be such a number that $x'' - r$ is less than 1: then, x'' is less than $r + 1 = r + \dfrac{1}{x'''}$, suppose, where x''' is greater than 1:

$$\therefore\ x = p + \cfrac{1}{q + \cfrac{1}{r + \cfrac{1}{x'''}}}:$$

and by a continuation of the process, it is evident that the surd will be expressed in the proposed form, to any extent that may be required.

Ex. 1. Express $\sqrt{2}$ in the form of a continued fraction.

Here, $\sqrt{2} = 1 + \dfrac{\sqrt{2} - 1}{1} = 1 + \dfrac{1}{\sqrt{2} + 1}$:

$$\sqrt{2} + 1 = 2 + \frac{\sqrt{2} - 1}{1} = 2 + \frac{1}{\sqrt{2} + 1}:$$

so that the values of x', x'', x''', &c. are all equal to $\sqrt{2}+1$, and those of q, r, &c. to 2:

$$\therefore \sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \text{\&c. } \textit{in infinitum}.}}$$

To find the successive converging fractions, we have

$$\begin{array}{cccccccc} & 1, & 2, & 2, & 2, & 2, & \text{\&c.} \\ \frac{1}{0}: & \frac{1}{1}, & \frac{3}{2}, & \frac{7}{5}, & \frac{17}{12}, & \frac{41}{29}, & \text{\&c.}: \end{array}$$

and these are all in their lowest terms: are alternately less and greater than the true value: and become successively more and more nearly equal to it.

Ex. 2. Find the fractions converging to the value of $\sqrt{15}$.

Here, $\sqrt{15} = 3 + \dfrac{\sqrt{15}-3}{1} = 3 + \dfrac{6}{\sqrt{15}+3} = 3 + \cfrac{1}{\cfrac{\sqrt{15}+3}{6}}$:

$$\frac{\sqrt{15}+3}{6} = 1 + \frac{\sqrt{15}-3}{6} = 1 + \frac{1}{\sqrt{15}+3}:$$

$$\sqrt{15}+3 = 6 + \frac{\sqrt{15}-3}{1} = 6 + \frac{6}{\sqrt{15}+3} = 6 + \cfrac{1}{\cfrac{\sqrt{15}+3}{6}}:$$

and the values of x', x'', &c. afterwards recur:

$$\therefore \sqrt{15} = 3 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{1 + \cfrac{1}{6 + \text{\&c. } \textit{in infinitum}.}}}}$$

Hence, to find the converging fractions, we have

$$\begin{array}{cccccccc} & 3, & 1, & 6, & 1, & 6, & \text{\&c.} \\ \frac{1}{0}: & \frac{3}{1}, & \frac{4}{1}, & \frac{27}{7}, & \frac{31}{8}, & \frac{213}{55}, & \text{\&c.} \end{array}$$

Ex. 3. To exhibit $\sqrt{19}$ in the form of a continued fraction.

Here,

$$\sqrt{19} = 4 + \frac{\sqrt{19}-4}{1} = 4 + \frac{3}{\sqrt{19}+4} = 4 + \frac{1}{\frac{\sqrt{19}+4}{3}}:$$

$$\frac{\sqrt{19}+4}{3} = 2 + \frac{\sqrt{19}-2}{3} = 2 + \frac{5}{\sqrt{19}+2} = 2 + \frac{1}{\frac{\sqrt{19}+2}{5}}:$$

$$\frac{\sqrt{19}+2}{5} = 1 + \frac{\sqrt{19}-3}{5} = 1 + \frac{2}{\sqrt{19}+3} = 1 + \frac{1}{\frac{\sqrt{19}+3}{2}}:$$

$$\frac{\sqrt{19}+3}{2} = 3 + \frac{\sqrt{19}-3}{2} = 3 + \frac{5}{\sqrt{19}+3} = 3 + \frac{1}{\frac{\sqrt{19}+3}{5}}:$$

$$\frac{\sqrt{19}+3}{5} = 1 + \frac{\sqrt{19}-2}{5} = 1 + \frac{3}{\sqrt{19}+2} = 1 + \frac{1}{\frac{\sqrt{19}+2}{3}}:$$

$$\frac{\sqrt{19}+2}{3} = 2 + \frac{\sqrt{19}-4}{3} = 2 + \frac{1}{\sqrt{19}+4} = 2 + \frac{1}{\frac{\sqrt{19}+4}{1}}:$$

$$\frac{\sqrt{19}+4}{1} = 8 + \frac{\sqrt{19}-4}{1} = 8 + \frac{3}{\sqrt{19}+4} = 8 + \frac{1}{\frac{\sqrt{19}+4}{3}}:$$

and after this, the quotients 2, 1, 3, 1, 2, 8, will evidently recur continually in the same order: and the continued and converging fractions are thus easily exhibited.

336. We will conclude this chapter by shewing in what manner the use of continued fractions is rendered available in the solution of indeterminate equations of the first degree, as stated in article (320).

337. To make the solution of the equation $ax - by = 1$, to depend upon that of $ap - bq = -1$.

Here, $1 = -ap + bq = abr - ap - abr + bq$

$$= a(rb - p) - b(ra - q):$$

$$\therefore a(rb - p) - b(ra - q) = 1:$$

and all the solutions of the equation will be comprised in

$$x = rb - p, \text{ and } y = ra - q,$$

where r may be assumed at pleasure.

338. To make the solution of the equation $ax - by = -1$, dependant upon that of $ap - bq = 1$.

Here, $-1 = -ap + bq = abr - ap - abr + bq$

$$= a(rb - p) - b(ra - q):$$

$$\therefore a(rb - p) - b(ra - q) = -1:$$

and all the solutions of the equation will be comprised in the same formulæ as before.

339. To solve the equation $ax - by = \pm 1$, in whole numbers.

Let the fractions successively converging to the value of $\frac{b}{a}$, be $\frac{p_1}{q_1}$, $\frac{p_2}{q_2}$, &c. $\frac{p}{q}$: then, we shall have in all cases,

$$ap - bq = \pm 1:$$

whence, by the preceding articles, all the values of x and y will be immediately obtained.

Ex. 1. Solve $15x - 17y = 1$, in whole numbers.

Here, the converging fractions are $\frac{1}{1}$, $\frac{8}{7}$, $\frac{17}{15}$:

$$\therefore 15p - 17q = 1:$$

and one solution will manifestly be $x = 8$, $y = 7$: also, the general solutions will be

$$x = 8 \pm 17r, \text{ and } y = 7 \pm 15r,$$

where r is any whole number whatever.

Here, the least positive values of x and y are 8 and 7.

Ex. 2. Solve $19x - 10y = 1$, in whole numbers.

Here, the converging fractions are $\frac{1}{1}$, $\frac{2}{1}$, $\frac{19}{10}$:

$$\therefore\ 10p - 19q = 1, \text{ or } 19q - 10p = -1:$$

$$\text{but } 1 = -19q + 10p = 19(10r - q) - 10(19r - p):$$

$$\therefore\ 19(10r - 1) - 10(19r - 2) = 1:$$

and all the solutions of the equation $19x - 10y = 1$, are comprised in $x = 10r - 1$, and $y = 19r - 2$, where r may be assumed at pleasure.

Here, the least positive values of x and y are 9 and 17.

Ex. 3. Solve the equation $9x - 13y = -1$, in whole numbers.

Here, the converging fractions are $\frac{1}{1}$, $\frac{3}{2}$, $\frac{13}{9}$:

$$\text{therefore, } 9p - 13q = 1:$$

$$\text{but } -1 = -9p + 13q = 9(13r - p) - 13(9r - q):$$

$$\therefore\ 9(13r - 3) - 13(9r - 2) = -1:$$

and the general values of x and y will be expressed by

$$x = 13r - 3, \text{ and } y = 9r - 2.$$

Here, if $r = 1$, we have $x = 10$ and $y = 7$, the least numbers satisfying the equation.

340. To solve the equation $ax - by = \pm c$, in whole numbers.

Let $\frac{p}{q}$ be the last of the fractions converging to the value of $\frac{b}{a}$: then, we have

$$ap - bq = \pm 1:$$

$$\therefore\ acp - bcq = \pm c: \text{ also, } abr - abr = 0:$$

$$\text{whence, } a(cp \mp br) - b(cq \mp ar) = \pm c:$$

and therefore $x = cp \mp rb$, and $y = cq \mp ra$, are the general values of x and y.

Also, for positive integral values of x and y, we must have r less than either $\frac{cp}{b}$ or $\frac{cq}{a}$, when the negative sign is used.

Ex. Solve $7x - 12y = 19$, in whole numbers.

Here, the converging fractions are $\frac{1}{1}$, $\frac{2}{1}$, $\frac{5}{3}$, $\frac{12}{7}$:

$$\therefore\ 7p - 12q = -1:$$

$$\therefore\ 7.19p - 12.19q = -19:$$

$$\therefore\ -7.19p + 12.19q = 19, \text{ and } 7.12r - 7.12r = 0:$$

$$\text{whence, } 7(12r - 19p) - 12(7r - 19q) = 19:$$

and $x = 12r - 95$, and $y = 7r - 57$, are the general values required, that admit of no limitation.

In this instance, for positive whole numbers, r must not be less than $\frac{95}{12}$ or $\frac{57}{7}$: and therefore the least value of r will be 9, and those of x and y are 13 and 6 respectively.

341. To solve the equation $ax + by = c$, in whole numbers.

Let $\frac{p}{q}$ be the last of the convergents to $\frac{b}{a}$: then, as before,

$$ap - bq = \pm 1:$$

$$\therefore\ \pm acp \mp bcq = c: \text{ also, } abr - abr = 0:$$

$$\text{whence, } a(\pm cp - br) + b(ar \mp cq) = c:$$

and therefore the values satisfying the equation will be

$$x = \pm cp - br \text{ and } y = ra \mp cq:$$

and for positive values, the limits of r may be found as before.

Ex. 1. Solve $11x + 5y = 1031$, in whole numbers.

Here, the converging fractions are $\frac{2}{1}$, $\frac{11}{5}$:

$$\therefore\ 11p - 5q = 1:$$

$$\therefore\ 11.1031\,p - 5.1031\,q = 1031:$$

$$\text{also,}\ 11.5r - 11.5r = 0:$$

$$\therefore\ 11(1031p - 5r) + 5(11r - 1031q) = 1031.:$$

$$\text{and}\ x = 1031 - 5r,\quad y = 11r - 2062.$$

For positive values of x and y we must evidently have r less than $\frac{1031}{5} = 206\frac{1}{5}$, and greater than $\frac{2062}{11} = 187\frac{5}{11}$: and the number of solutions will therefore be $206 - 187 = 19$, agreeably to article (302).

Ex. 2. In how many different ways may £100. be paid in crowns and moidores?

Let x and y denote the numbers of crowns and moidores respectively:

then, $5x + 27y = 2000$, by the question:

and the converging fractions are $\frac{5}{1}$, $\frac{11}{2}$, $\frac{27}{5}$:

that is, $p = 11$ and $q = 2$:

$\therefore$ from article (302), the difference of the integral parts of $\frac{cp}{b}$ and $\frac{cq}{a}$ is 14, which is the number of ways required.

342. Approximate to the value of $\sqrt{5}$, by means of a continued fraction.

Here, $$\sqrt{5} = 2 + \frac{\sqrt{5} - 2}{1} = 2 + \frac{1}{\sqrt{5} + 2}:$$

$$\sqrt{5} + 2 = 4 + \frac{\sqrt{5} - 2}{1} = 4 + \frac{1}{\sqrt{5} + 2}:$$

and since $\sqrt{5}+2$ now continually recurs, we have the quotients 2, 4, 4, &c.: and the corresponding converging fractions will be

$$\frac{2}{1},\ \frac{9}{4},\ \frac{38}{17},\ \frac{161}{72},\ \frac{682}{305},\ \frac{2889}{1292},\ \&c.:$$

$\therefore \sqrt{5}$ is greater than $\dfrac{682}{305}$ and less than $\dfrac{2889}{1292}$: and because $\dfrac{2889}{1292}-\sqrt{5}$ is less than $\sqrt{5}-\dfrac{682}{305}$, and $\therefore 2\left(\dfrac{2889}{1292}\right)-2\sqrt{5}$ is less than $\dfrac{2889}{1292}-\dfrac{682}{305}$: or $\dfrac{2889}{1292}-\sqrt{5}$ is less than $\dfrac{1}{2}\left(\dfrac{2889}{1292}-\dfrac{682}{305}\right)$: it differs from the latter, by a quantity less than $\dfrac{1}{2}\left(\dfrac{2889}{1292}-\dfrac{682}{305}\right)$, or less than $\dfrac{1}{2\times 305\times 1292}$.

343. The ratio of the area of a regular decagon described about a circle to that of another inscribed in the circle, is expressed by $\dfrac{8}{5+\sqrt{5}}$: find its approximate values.

Proceeding as in the last article, we find the ratio to be expressed by the continued fraction

$$\cfrac{8}{7+\cfrac{1}{4+\cfrac{1}{4+\&c.}}}$$

and for the denominator of this fraction, the quotients are

$$7,\quad 4,\quad 4,\quad 4,\quad 4,\quad \&c.:$$

$$\text{and}\quad \frac{1}{0}:\ \frac{7}{1},\ \frac{29}{4},\ \frac{123}{17},\ \frac{521}{72},\ \frac{2107}{305},\ \&c.:$$

$\therefore$ the successive approximations required are

$$\frac{8}{7},\ \frac{32}{29},\ \frac{136}{123},\ \frac{576}{521},\ \frac{2440}{2107},\ \&c.$$

344. To represent x in the form of a continued fraction, from the equation $x^2 - px + q = 0$.

Here, $x^2 = px - q$, and by dividing by x and successively substituting for its value, we have

$$x = \frac{px - q}{x} = p - \frac{q}{x} = p - \cfrac{q}{p - \cfrac{q}{x}}$$

$$= p - \cfrac{q}{p - \cfrac{q}{p - \&c.}}$$

This is not a continued fraction of the kind we have been discussing, inasmuch as each of the numerators is not $= 1$, nor are the signs positive, as has hitherto been supposed: it will however enable us, in certain cases, to find an approximate value of a root: thus, if q be very small compared to p, we have $x = p$, nearly, as a first approximation:

$$x = p - \frac{q}{p} = \frac{p^2 - q}{p}, \text{ nearly,}$$

as a second approximation: &c. = &c.

CHAPTER XIV.

SCALES OF NOTATION.

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345. DEF. NOTATION is the method of representing abstract numerical magnitudes by means of symbols: and it comprises different *Scales* dependent upon the numbers of the symbols or figures employed.

346. *If* r *be any whole number, and* a_0, a_1, a_2, *&c.*, a_m *be integers less than* r, *any number whatever may be represented in the form:*

$$N = a_m r^m + a_{m-1} r^{m-1} + a_{m-2} r^{m-2} + \&c. + a_2 r^2 + a_1 r + a_0 .$$

For, let N be divided by the highest power of r contained in it, as r^m, the quotient being a_m, and the remainder N_1: then, we shall have

$$N = a_m r^m + N_1:$$

again, let N_1 be divided by the highest power of r contained in it, as r^{m-1}, and let the quotient and remainder be a_{m-1} and N_2 respectively:

$$\therefore\ N_1 = a_{m-1} r^{m-1} + N_2:$$

$$\text{whence,}\quad N = a_m r^m + N_1 = a_m r^m + a_{m-1} r^{m-1} + N_2: \ \&c.$$

and, continuing this operation till the remainder becomes less than r, we shall at last come to the form,

$$N = a_m r^m + a_{m-1} r^{m-1} + a_{m-2} r^{m-2} + \&c. + a_2 r^2 + a_1 r + a_0:$$

which, by reversing the order of the terms, may be written,

$$N = a_0 + a_1 r + a_2 r^2 + \&c. + a_{m-2} r^{m-2} + a_{m-1} r^{m-1} + a_m r^m:$$

and because in each successive division, the highest possible power of r is taken, it follows that each of the numbers a_0, a_1, a_3, &c., a_m is less than r.

347. Def. The number represented by the symbol r in the last article, is called the *Radix* or *Base* of the scale of notation: and the numbers a_0, a_1, a_2, &c. a_m, are termed the *Digits* belonging to the scale of notation whose radix is r.

348. Cor. 1. Hence, a number consisting of p figures or digits, may be expressed in the form,

$$N = a_0 + a_1 r + a_2 r^2 + \&c. + a_{p-3} r^{p-3} + a_{p-2} r^{p-2} + a_{p-1} r^{p-1}:$$

and the greatest and least numbers consisting of p digits will therefore be $r^p - 1$ and r^{p-1} respectively.

349. Cor. 2. If the order of the digits of N be inverted, and the resulting number be denoted by n, we shall have

$$N = a_m r^m + a_{m-1} r^{m-1} + \&c. + a_1 r + a_0:$$

$$n = a_0 r^m + a_1 r^{m-1} + \&c. + a_{m-1} r + a_m:$$

whence, subtracting the latter from the former, and arranging the difference according to descending powers of r, we shall have

$$N - n = a_m (r^m - 1) + a_{m-1} r (r^{m-2} - 1) + a_{m-2} r^2 (r^{m-4} - 1) + \&c.,$$

which, by article (35), is universally divisible by $r - 1$.

350. Def. When the value of r is 2, 3, 4, &c. the scale of notation is termed the *Binary*, *Ternary*, *Quaternary*, &c.: and because the remainders may be any numbers *less* than r; 0, 1, 2, 3, 4, &c. $r - 1$ will be the *Digits*, which can neither be *more* nor *fewer* than the radix of the scale.

Thus, in the senary scale, $453 = 4.6^2 + 5.6 + 3$: in the duodenary scale, $3807 = 3.12^3 + 8.12^2 + 0.12 + 7$.

From these examples it appears that in the expression of any number, every digit, in addition to its *original* and *natural* value, possesses also a *local* value, which depends

upon the place it occupies and the radix of the scale to which it belongs: and the digits are sometimes styled so many units of the *first*, *second*, *third*, &c., *orders*, according as they are found in the first, second, third, &c. places from the right hand.

Thus, in the common or denary scale, 2079 denotes 2000, 70 and 9, the values of the figures 2 and 7 depending entirely upon their situations.

351. In any scale of notation whose radix is r, a number N when divided by $r-1$, leaves the same remainder as the sum of its digits, when divided by $r-1$, leaves.

For, let
$$\begin{aligned} N &= a + br + cr^2 + dr^3 + \&c. \\ &= b(r-1) + c(r^2-1) + d(r^3-1) + \&c. \\ &\quad + a + b + c + d + \&c.: \end{aligned}$$
then, because each of the factors $r-1$, r^2-1, r^3-1, &c. is divisible by $r-1$, it follows that N and $a+b+c+d+$ &c. when divided by $r-1$, leave the same remainder.

Hence, if the sum of the digits of a number be divisible by $r-1$, the number itself is so too: and if from any number the sum of its digits be subtracted, the remainder is always divisible by $r-1$.

In the common scale of notation, a number is always divisible by 9, when the sum of its digits is divisible by 9: and any number and the sum of its digits, when divided by 9, leave the same remainder.

352. Cor. From this property, is derived a test of the accuracy of the operation of Multiplication, by *casting out the nines*.

Let A and B contain p and q nines respectively, with the remainders α, β: so that
$$A = 9p + \alpha, \quad B = 9q + \beta:$$
$$\begin{aligned} \text{then, } AB &= (9p+\alpha)(9q+\beta) \\ &= 81pq + 9q\alpha + 9p\beta + \alpha\beta \\ &= 9(9pq + q\alpha + p\beta) + \alpha\beta: \end{aligned}$$

therefore, AB and $\alpha\beta$, when divided by 9, leave the same remainder: that is, the sum of the digits of the product, when divided by 9, leaves the same remainder, as the product of the partial remainders leaves.

Ex. If $A = 27354$ and $B = 2687$: then, it is easily found that $\alpha = 3$ and $\beta = 5$:

$$\text{also, } AB = 73500198, \text{ and } \alpha\beta = 15:$$

and it is seen immediately that the remainders arising from the division of the sums of the digits in AB and $\alpha\beta$ by 9 are both equal to 6: from which it is inferred that the multiplication has been correctly performed: and it can be erroneous only by some multiple of 9, or in the placing of its different parts.

353. In any scale of notation, whose radix is r, the difference of the sums of the digits in the odd and even places, when divided by $r+1$, leaves the same remainder as the number N, when divided by $r+1$, leaves.

$$\begin{aligned} \text{Let } N &= a + br + cr^2 + dr^3 + \&c. \\ &= b(r+1) + c(r^2-1) + d(r^3+1) + \&c. \\ &\quad + a - b + c - d + \&c.: \end{aligned}$$

then, since $r^m + 1$ is always divisible by $r+1$ when m is odd, and $r^m - 1$ is divisible by $r+1$, when m is even, it follows that the remainders arising from the division of N and $a-b+c-d+$ &c. by $r+1$, will be the same.

Hence, also, we have $N - (a + c + \&c.) + (b + d + \&c.)$

$$= b(r+1) + c(r^2-1) + d(r^3+1) + \&c.:$$

so that if to any number there be added, the excess of the sum of the digits in the even places above the sum of those in the odd places, the result will be divisible by $r+1$.

If $a + c + \&c. = b + d + \&c.$, it is evident that N is divisible ...ary scale, the common scale of notation, any number

From these exa... 1, if the sum of the *first*, *third*, &c. digits of any number, every d... any multiple of 11 from, the sum of *natural* value, possesses ...igits.

354. From the nature of the scales, as above explained, it is evident that all numbers represented in them, may be transformed to the denary or common scale, by merely performing the operations implied.

Thus, 234 in the quinary scale is equivalent to

$$2.5^2 + 3.5 + 4 = 50 + 15 + 4 = 69,$$

in the common scale: and this equality may be written in the form $(234)_5 = (69)_{10}$.

Ex. Two hundred and fifteen is expressed by 425, in a certain scale: required the radix.

Let r denote the radix required: then, we have

$$4r^2 + 2r + 5 = 215, \text{ or } r^2 + \tfrac{1}{2}r = 52\tfrac{1}{2}:$$

whence, by the solution of the quadratic, we find $r = 7$ and $r = -7\frac{1}{2}$, the latter of which is excluded by the circumstance of not admitting either fractional or negative bases: and therefore 7 is the radix sought, as may easily be verified.

355. *To perform the arithmetical operations of Addition, Subtraction, &c. in a scale of notation, whose radix is* r.

From what has been already said, and from the nature of the proposed operations, it is obvious that the processes will be similar to those used in the common scale of notation, with this difference only, that r must be used in the cases wherein the number 10 would be applied, did the numbers proposed belong to the common scale: this will be understood by means of the following examples.

Ex. 1. Find the sum and difference of 45324502 and 25405534 in the senary scale, or scale whose radix is 6.

First, arranging the numbers as in common arithmetic,

$$\begin{array}{r} 45324502 \\ 25405534 \\ \hline \end{array}$$

the sum = 115134440,

which is obtained by adding the numbers in vertical lines, carrying 1 for every 6 contained in the results, and putting down the excesses above it:

$$\begin{array}{rr} & 45324502 \\ & 25405534 \\ \cline{2-2} \text{the difference} = & 15514524, \end{array}$$

which is found by subtraction, where we always borrow 6, when the digit in the lower line exceeds that in the upper, and add 1 to the next digit in the lower line for it.

Ex. 2. Multiply 2483 by 589 in the undenary scale, or scale whose base is 11.

$$\begin{array}{rr} \text{Here, we have} & 2483 \\ & 589 \\ \cline{2-2} & 1t985 \\ & 18502 \\ & 11184 \\ \cline{2-2} \text{the product} = & 13122t5, \end{array}$$

which is obtained as in ordinary multiplication, by carrying 1 at every 11, the letter t being here supposed to represent 10, because we have no single arithmetical symbol to denote it.

Ex. 3. Divide 1184323 by 589 in the duodenary scale, whose base is 12.

$$\begin{array}{rl} \text{Here, we have} & 589)\,1184323\,(2483 \\ & u56 \\ \cline{2-2} & 22t3 \\ & 1tu0 \\ \cline{2-2} & 3u32 \\ & 39t0 \\ \cline{2-2} & 1523 \\ & 1523 \\ \cline{2-2} \end{array}$$

where t and u represent 10 and 11: and the operation is conducted with reference to 12, as it is ordinarily done with respect to 10.

Ex. 4. Involution and Evolution are performed in a manner precisely similar: thus, in the senary scale, we shall find

$$(2405)^2 = 2405 \times 2405 = 11122441:$$

and to extract the square root of 11122441, we have, after pointing as in common arithmetic,

$$\begin{array}{r}
\dot{1}1\dot{1}2\dot{2}4\dot{4}1\,(2405 \\
4\, \\
\hline
44)\,312\, \\
304\, \\
\hline
5205)\,42441\, \\
42441\, \\
\hline
\end{array}$$

356. Cor. If $N = a_m r^m + a_{m-1} r^{m-1} + \&c. + a_1 r + a_0$, and both members be multiplied by r^n, we shall have

$$r^n N = a_m r^{m+n} + a_{m-1} r^{m+n-1} + \&c. + a_1 r^{n+1} + a_0 r^n:$$

where each of the last n digits is 0: in other words, a number may be multiplied by any power of the radix, by affixing to it as many ciphers as there are units in its index: and conversely.

357. *From a number expressed in a scale whose radix is* r, *to find the digits expressing it in a scale whose radix is* ρ.

Let N be the given number, and suppose

$$N = a_m \rho^m + a_{m-1} \rho^{m-1} + \&c. + a_2 \rho^2 + a_1 \rho + a_0,$$

where the values of a_m, a_{m-1}, &c., a_2, a_1, a_0, are to be determined:

then we observe, that if N be divided by the new radix ρ, the remainder will be a_0, the digit in the units' place: again,

if the last quotient be divided by ρ, the remainder a_1 is the second digit: and the same will hold for succeeding digits; and the requisite operation will therefore be manifest, as it appears that the required digits, beginning at the right hand, are the remainders after the successive divisions of the number, by the radix of the new scale proposed.

Ex. 1. Express the common number 75432, in the senary and duodenary scales.

In the former case $\rho = 6$, and in the latter $\rho = 12$: whence we have the following operations:

$$\begin{array}{rl}
6)\underline{75432} & \\
6)\underline{12572} & 0 = a_0, \\
6)\underline{2095} & 2 = a_1, \\
6)\underline{349} & 1 = a_2, \\
6)\underline{58} & 1 = a_3, \\
6)\underline{9} & 4 = a_4, \\
6)\underline{1} & 3 = a_5, \\
0 & 1 = a_6:
\end{array}
\qquad
\begin{array}{rl}
12)\underline{75432} & \\
12)\underline{6286} & 0 = a_0, \\
12)\underline{523} & t = a_1, \\
12)\underline{43} & 7 = a_2, \\
12)\underline{3} & 7 = a_3, \\
0 & 3 = a_4:
\end{array}$$

and the common number 75432 is expressed in the senary scale by 1341120, and in the duodenary scale by $377t0$: and it will follow generally, that the greater the radix of the scale proposed, the less will be the number or magnitude of the digits requisite to express a given number.

Ex. 2. Convert 3256 from a scale whose radix is 7, to one whose local value is 12.

Here,
$$\begin{array}{rl}
12)\underline{3256} & \\
12)\underline{166} & 4 = a_0, \\
12)\underline{11} & 1 = a_1, \\
0 & 8 = a_2:
\end{array}$$

whence the required expression is 814; and the divisions by 12 have been conducted by means of the local value 7.

358. To avoid the operation of division in scales, to which we are not accustomed, it will generally be found convenient, first to transform the number whose radix is r to the common scale, by article (354): and then to convert the result into the scale whose radix is ρ.

Thus, in the last example, $(3256)_7 = (1168)_{10}$: and $(1168)_{10} = (814)_{12}$, by the last article.

359. Every number whatever, is composed of the sum of certain terms of the geometrical series, 1, 2, 2^2, 2^3, &c.

For, in the binary scale of notation, we have

$$N = a_m 2^m + a_{m-1} 2^{m-1} + \&c. + a_2 2^2 + a_1 2 + a_0,$$

where N may be any number whatever, and

$$a_m, \ a_{m-1}, \ \&c., \ a_2, \ a_1, \ a_0,$$

are each less than 2, and must therefore be either 0 or 1: that is, none of the terms of the progression are taken more than *once*, and consequently all numbers whatever may be composed out of the sum of them, by assigning proper values to m.

Ex. Express 37 by means of the terms of the series,

1, 2, 2^2, 2^3, &c.

Here, to transform 37 into the binary scale, we have

$$\begin{array}{rl}
2\,)\,\underline{37} & \\
2\,)\,\underline{18} & 1 = a_0, \\
2\,)\,\underline{9} & 0 = a_1, \\
2\,)\,\underline{4} & 1 = a_2, \\
2\,)\,\underline{2} & 0 = a_3, \\
2\,)\,\underline{1} & 0 = a_4, \\
0 & 1 = a_5:
\end{array}$$

whence, the number 37 is equivalent to 100101 in the binary scale, which is expressed by $2^5 + 2^2 + 1$.

Thus, out of a set of weights of 1 lb., 2 lbs., 4 lbs., &c., it will be necessary to select 32 lbs., 4 lbs., and 1 lb., in order to weigh a substance of 37 lbs.

360. All numbers whatever may be formed, by the sums and differences of certain terms of the geometrical progression 1, 3, 3^2, 3^3, &c.

For, in the ternary scale of notation, we have

$$N = a_m 3^m + a_{m-1} 3^{m-1} + \&c. + a_2 3^2 + a_1 3 + a_0,$$

in which each of the coefficients a_m, a_{m-1}, &c., a_2, a_1, a_0 being less than 3, must manifestly be 2, 1 or 0.

If every one of the coefficients be 0 or 1, the proposition is evident: but if one or more of them be 2, as when

$$N = 2.3^m + 2.3^{m-1} + \&c + 2.3^2 + 2.3 + 2:$$

$$\begin{aligned} \text{then,}\quad 2.3^m &= (3-1)\,3^m = 3^{m+1} - 3^m: \\ 2.3^{m-1} &= (3-1)\,3^{m-1} = 3^m - 3^{m-1}; \\ \&c. &= \quad \&c. \quad = \quad \&c. \\ 2.3^2 &= (3-1)\,3^2 = 3^3 - 3^2: \\ 2.3 &= (3-1)\,3 = 3^2 - 3: \\ 2 &= 3-1 \quad = 3-1: \end{aligned}$$

$$\text{and} \therefore \text{by addition, } N = 3^{m+1} - 1:$$

and a similar proceeding may be adopted in every other case.

Ex. Which of the series of weights, 1 lb., 3 lbs, 9 lbs., &c. must be selected to weigh 206 lbs.?

By the general method, we have 206 expressed in the ternary scale by $21122 = 2.3^4 + 1.3^3 + 1.3^2 + 2.3 + 2$; where it would be requisite to use more than one weight of the same kind: and this may be obviated as follows:

$$\begin{array}{rr}
206 \text{ is equivalent to } & 21122 \\
1 & 1 \\
\hline
206+1 & 21200 \\
9 & 100 \\
\hline
206+1+9 & 22000 \\
27 & 1000 \\
\hline
206+1+9+27 & 100000 = 243:
\end{array}$$

$$\text{whence, } 206 = 243 - 27 - 9 - 1 = 3^5 - (3^3 + 3^2 + 1):$$

that is, the problem will be solved, by putting 3^5 into one scale: and $3^3 + 3^2 + 1$ into the other scale, with the substance weighed.

Similar considerations will sometimes enable us to obtain analogous results in other scales.

361. Def. Since the local value of every digit increases *r-fold* as we advance towards the left hand, if the radix of the scale be r, it will follow that if the digits be taken in the contrary order, their local values must decrease in the same proportion.

Hence therefore, the local value of each of the digits in succession to the right of the units' place becomes r times *less* than that of the one which immediately precedes it: that is, if $N = a_m r^m + a_{m-1} r^{m-1} + \&c. + a_1 r + a_0 r^0 + a_{-1} r^{-1} + a_{-2} r^{-2} + \&c.$, the quantities to the left of $a_0 r^0$ or a_0, comprising units of orders *superior* to the first, will be whole numbers, whilst those to the right of the same term, being of local values *inferior* to the first, designate so many fractions: and in a quantity consisting of both, it is usual to separate the integral part from that which is fractional, by means of a point.

Thus, in the ternary scale, 120.21 is equivalent to

$$1 \cdot 3^2 + 2 \cdot 3 + 0 \cdot 3^0 + 2 \cdot 3^{-1} + 1 \cdot 3^{-2}$$

$$= 1 \cdot 3^2 + 2 \cdot 3 + 0 + \frac{2}{3} + \frac{1}{3^2}.$$

362. Cor. 1. Hence, in order to multiply or divide a number by any power of the radix, we have only to remove the separating point towards the right or left, as many places as there are units contained in its index, since by such a step, the denomination of every digit is increased or diminished in the proposed ratio.

363. Cor. 2. Every quantity of this description, may readily be expressed in the form of a vulgar fraction.

$$\text{For, } N = a_{-1}r^{-1} + a_{-2}r^{-2} + a_{-3}r^{-3} + \&c. + a_{-m}r^{-m}$$

$$= \frac{a_{-1}}{r} + \frac{a_{-2}}{r^2} + \frac{a_{-3}}{r^3} + \&c. + \frac{a_{-m}}{r^m}$$

$$= \frac{a_{-1}r^{m-1} + a_{-2}r^{m-2} + a_{-3}r^{m-3} + \&c. + a_{-m}}{r^m}:$$

from which we conclude that any quantity consisting of m digits to the right of the separating point, may be represented by a vulgar fraction whose numerator is the said collection of digits considered integral, and the denominator the m^{th} power of the radix, or 1 followed by m ciphers: and conversely.

$$\text{Thus, } (.324)_6 = \frac{3}{6} + \frac{2}{6^2} + \frac{4}{6^3} = \frac{3.6^2 + 2.6 + 4}{6^3} = \left(\frac{324}{1000}\right)_6:$$

$$\text{also, } \left(\frac{798}{100000}\right)_{11} = (.00798)_{11}, \ \&c.$$

364. Cor. 3. A vulgar fraction, whose denominator is *not* 1, followed by a number of ciphers, may be converted into the form of a whole number, by means of the same principles.

$$\text{Thus, } \left(\frac{54}{300}\right)_7 = \left(\frac{54}{3 \times 100}\right)_7 = (\tfrac{1}{3})_7 (.54)_7 = (.16)_7:$$

from which we infer that a vulgar fraction may always be represented after the manner of whole numbers, by affixing to the numerator as many ciphers as may be necessary, and then effecting the division by the denominator.

The cipher may be affixed to the numerator as it stands: or, when the division has been effected as far as the units' place,

the remainder being then less than the divisor, if we affix a cipher to it, we reduce it to units of the next inferior order, and this will also be the denomination of the digit then obtained for the quotient: and so on.

365. Cor. 4. Should the division never terminate, but the digits continually recur in the same order, the quotient is termed *periodical*, the figures which recur, being styled its *Period:* and the quantity is denominated a *simple* or *mixed* periodical fraction, according as the period commences at the first digit on the right of the separating point, or afterwards.

Thus, in the quinary scale of notation, we have

$$\tfrac{1}{3} = \tfrac{1}{3}\,(1.00000\ \&c.) = .131313\ \&c.$$

which is a simple periodical quantity: and in the denary scale,

$$\frac{101}{110} = \frac{1}{11}\,(10.1000\ \&c.) = .91818\ \&c.$$

which is a mixed periodical decimal.

366. Cor. 5. Every periodical quantity may be expressed exactly, by means of a vulgar fraction.

First, taking a simple periodical quantity, where each period consists of q digits in a scale whose radix is r, let us assume

$$\Sigma = .QQQ\ \&c.: \ \therefore\ r^q\,\Sigma = Q.QQQ\ \&c. \text{ by article (362):}$$

$$\text{whence, } (r^q - 1)\,\Sigma = Q, \text{ and } \therefore\ \Sigma = \frac{Q}{r^q - 1}.$$

Next, let the quantity be mixedly periodical, in which P and Q consist of p and q digits respectively: then, if $\Sigma = .PQQQ$ &c., we have

$$r^{p+q}\,\Sigma = PQ.QQQ\ \&c., \text{ and } r^p\,\Sigma = P.QQQ\ \&c.:$$

$$\text{whence, } (r^{p+q} - r^p)\,\Sigma = PQ - P, \text{ and } \therefore\ \Sigma = \frac{PQ - P}{r^p\,(r^q - 1)}.$$

Both these results are in the forms of finite vulgar fractions.

367. In the addition and subtraction of fractions thus expressed, it is manifest that in order to have the same denomina-

tions of units combined together, the points in all the quantities concerned must be in the same vertical line, and then the operation may be effected as in integers: but some additional considerations will be necessary to form a proper estimate, of the results of the operations of multiplication and division.

Let P, Q comprise p, q digits to the right of the separating point respectively: then, these quantities represented as vulgar fractions are $\frac{P}{r^p}$ and $\frac{Q}{r^q}$:

therefore their product $= \frac{P}{r^p} \times \frac{Q}{r^q} = \frac{PQ}{r^{p+q}}$, which has $p+q$ digits to the right of the separating point:

also, the quotient $= \frac{P}{r^p} \div \frac{Q}{r^q} = \frac{\left(\frac{P}{Q}\right)}{r^{p-q}}$, which has therefore $p-q$ digits to the right of that point.

368. *To transform a fraction expressed in a given scale, into one belonging to another given scale.*

Let the given number expressed in the scale whose radix is r, be denoted by N, and if ρ be the radix of the new scale, assume

$$N = \beta_1 \rho^{-1} + \beta_2 \rho^{-2} + \beta_3 \rho^{-3} + \&c. + \beta_m \rho^{-m}$$

$$= \frac{\beta_1}{\rho} + \frac{\beta_2}{\rho^2} + \frac{\beta_3}{\rho^3} + \&c. + \frac{\beta_m}{\rho^m},$$

from which the values of β_1, β_2, β_3, &c., β_m, are to be determined:

$$\text{now, } \rho N = \beta_1 + \frac{\beta_2}{\rho} + \frac{\beta_3}{\rho^2} + \&c. + \frac{\beta_m}{\rho^{m-1}}:$$

$$\rho^2 N = \beta_1 \rho + \beta_2 + \frac{\beta_3}{\rho} + \&c. + \frac{\beta_m}{\rho^{m-2}}:$$

$$\rho^3 N = \beta_1 \rho^2 + \beta_2 \rho + \beta_3 + \&c. + \frac{\beta_m}{\rho^{m-3}}: \ \&c.$$

and these results prove that the first, second, third, &c. digits reckoned from the separating point, are the integers in the products which arise from multiplying the fractional parts of N, ρN, $\rho^2 N$, &c. successively by ρ.

Ex. 1. Express the common magnitude .015625 in the octenary scale, whose radix is 8.

Here, we have $N = .015625$:

$$8N = 0.125000: \quad \therefore \beta_1 = 0:$$

$$8^2 N = 1.000000: \quad \therefore \beta_2 = 1:$$

whence .015625 in the denary scale is equivalent to .01 in the octenary scale: and this may easily be verified: for,

$$(.01)_8 = \frac{0}{8} + \frac{1}{8^2} = \frac{1}{64} = .015625.$$

Ex. 2. Transform 14.125 from the denary, to the duodenary scale of notation.

First, for the integral part, we have

$$\begin{array}{r|l} 12\,)\,14 & \\ \hline 12\,)\;\;1 & 2 = \alpha_0: \\ \hline 0 & 1 = \alpha_1: \end{array}$$

again, for the fractional part, we have

$$\begin{array}{rl} .125 & \\ 12 & \\ \hline 1.500 & \therefore \beta_1 = 1: \\ 12 & \\ \hline 6.000 & \therefore \beta_2 = 6: \end{array}$$

whence, the required duodenary expression will be 12.16, the correctness of which may be shewn as above.

Ex. 3. Represent $(.015625)_{10}$ in the nonary scale, whose base is 9.

Proceeding according to the article, we have

$$
\begin{array}{r l}
.015625 & \\
9 & \\
\hline
0.140625 & \therefore\ \beta_1 = 0: \\
9 & \\
\hline
1.265625 & \therefore\ \beta_2 = 1: \\
9 & \\
\hline
2.390625 & \therefore\ \beta_3 = 2: \\
9 & \\
\hline
3.515625 & \therefore\ \beta_4 = 3: \\
9 & \\
\hline
4.640625 & \therefore\ \beta_5 = 4: \\
9 & \\
\hline
5.765625 & \therefore\ \beta_6 = 5: \\
9 & \\
\hline
6.890625 & \therefore\ \beta_7 = 6: \\
9 & \\
\hline
8.015625 & \therefore\ \beta_8 = 8: \\
9 & \\
\hline
0.140625 & \therefore\ \beta_9 = 0:\ \text{\&c.}
\end{array}
$$

whence, $(.015625)_{10}$ is equivalent to $.\dot{0}123456\dot{8}$, a recurrin quantity, in the nonary scale, the points over 0 and 8 denotin the extent of the period.

The result of this example, which is curious, shews th a terminating fractional quantity in one scale may be equ valent to a non-terminating quantity in another scale: an in all these instances we observe that the digits are higher lower according as the radix is higher or lower, contrary the remark of article (357) for whole numbers.

369. Cor. To avoid multiplication in scales with whi we may not be very familiar, a fraction may be transform

from a scale whose radix is r, to another whose radix is ρ, by first expressing it in the denary scale, and then transforming the result into the scale, whose radix is ρ.

370. For the *Theory* and *Practice* of *Decimals*, the student is referred to the fifth chapter of the author's *Arithmetic*, where he will find the subject treated at considerable length: and we will notice here only one or two circumstances respecting the nature of *Recurring* Decimals.

371. Recurring decimals are in reality equivalent to indefinitely extended geometrical progressions, as has been seen in (8) of article (216).

Thus, for the value of $.PPP$ &c. *in infinitum*, where P consists of p digits, we have, by article (204),

$$\sigma = \frac{a}{1-r}:$$

and here, $a = \dfrac{P}{10^p}$, and $r = \dfrac{1}{10^p}$:

whence, $\sigma = \dfrac{P}{10^p} \div \left(1 - \dfrac{1}{10^p}\right) = P \div (10^p - 1) = \dfrac{P}{10^p - 1}$:

from which formula, the value in any particular case may be obtained.

Also, conversely, an expression of the form $\dfrac{P}{10^p - 1}$, where P consists of not more than p digits, is equivalent to a simple recurring decimal whose period comprises p digits.

For, if we divide 1 by $10^p - 1$, according to the rules of Algebra, the quotient will be

$$\frac{1}{10^p} + \frac{1}{10^{2p}} + \frac{1}{10^{3p}} + \text{\&c. } \textit{in infinitum}:$$

$$\therefore \frac{P}{10^p - 1} = \frac{P}{10^p} + \frac{P}{10^{2p}} + \frac{P}{10^{3p}} + \text{\&c.}:$$

of which every term is a *repeating period* or *repetend*, consisting of p digits: and therefore the whole is a recurring decimal.

Again, for the value of $.PQQQ$ &c. *in infinitum*, where P and Q contain p and q digits respectively, we have

$$\sigma = \frac{P}{10^p} + \frac{1}{10^p} (.QQQ \text{ \&c. } in\ infinitum)$$

$$= \frac{P}{10^p} + \frac{1}{10^p}\left(\frac{Q}{10^q - 1}\right) = \frac{1}{10^p}\left\{P + \frac{Q}{10^q - 1}\right\}$$

$$= \frac{1}{10^p}\left\{\frac{10^q P - P + Q}{10^q - 1}\right\} = \frac{1}{10^p}\left\{\frac{PQ - P}{10^q - 1}\right\}:$$

since the denomination of P is 10^q times as great as that of Q: and therefore when they are regarded as whole numbers, $10^q P + Q$ is equivalent to PQ, where no arithmetical *operation* is intended to be expressed.

Conversely, when P and Q comprise not more than p and q digits respectively, any expression of the form

$$\frac{PQ - P}{10^p(10^q - 1)} = \frac{P(10^q - 1) + Q}{10^p(10^q - 1)},$$

will be equivalent to a mixed recurring decimal, consisting of a portion of p digits which does not recur, and a recurring portion whose period contains q digits.

For, $$\frac{P(10^q - 1) + Q}{10^p(10^q - 1)} = \frac{P}{10^p} + \frac{Q}{10^p}\left(\frac{1}{10^q - 1}\right)$$

$$= \frac{P}{10^p} + \frac{Q}{10^p}\left(\frac{1}{10^q} + \frac{1}{10^{2q}} + \frac{1}{10^{3q}} + \text{\&c.}\right):$$

the first term of which occupies the first p places, and the second commencing after p places, furnishes a recurring decimal consisting of q digits.

372. The view of the scales of notation taken in the preceding pages has presented us with several curious and interesting results, but with our *established system* of *Decimal*

Notation, it will not afford us many *practical* Theorems which could not be arrived at by other and much simpler means.

In the *Duodenary* or *Duodecimal* scale, which, with certain modifications, is applied to the estimation of *Artificers'* work, there must be 12 digits or symbols used; and these are generally the ten common digits with the letters t and u denoting 10 and 11.

The Arithmetical Operations of Multiplication and Division in this scale will be evinced in the following practical examples.

Ex. 1. Required the product of 9 feet 8 inches 7 parts, and 3 feet 10 inches.

Here, the number connecting the different parts of these quantities is 12, and expressed in the duodenary scale, they are 9.87 and $3.t$: whence, multiplying them together in this scale, we have

$$\begin{array}{r} 9.87 \\ 3.t \\ \hline 811t \\ 2519 \\ \hline 31.2tt \end{array}$$

and in this result $(31)_{12}$ represents feet, 2 denotes so many *twelfths* of a foot called *primes*, the former 10 so many *twelfths* of a prime termed *seconds*, and the latter 10 so many *twelfths* of a second, or 10 *thirds*: and it may be immediately converted into the common scale and nomenclature: for $(31)_{12} = (37)_{10}$:

$$\text{and } \therefore\ (31.2tt)_{12} = 37 + \frac{2}{12} + \frac{10}{12^2} + \frac{10}{12^3} \text{ square feet}$$

$$= 37 + \frac{34}{12^2} + \frac{120}{12^4} \text{ square feet}$$

= 37 square feet, 34 square inches, and 120 square parts.

Ex. 2. Divide 1532 feet $9\frac{9}{12}$ inches, by 81 feet 9 inches.

$$\text{Here, } 1532\text{ft. } 9\tfrac{9}{12}\text{in.} = (t78.99)_{12}:$$
$$\text{and, } 81\text{ft. } 9\text{in.} = (69.9)_{12}:$$

$$\begin{array}{l} \text{whence, } 69.9)\,t78.99\,(16.9 \\ \quad\quad\quad\quad\quad 699 \\ \quad\quad\quad\quad\quad \overline{39u9} \\ \quad\quad\quad\quad\quad 34t6 \\ \quad\quad\quad\quad\quad \overline{\;5139} \\ \quad\quad\quad\quad\quad \;5139 \\ \quad\quad\quad\quad\quad \overline{\quad\quad} \end{array}$$

that is, the quotient $= (16.9)_{12} = 18 + \dfrac{9}{12}$ feet $= 18$ feet 9 inches.

Ex. 3. Required the square root of 763 feet 1 inch 8 parts and 3 seconds.

Here, 763ft. 1in. 8pts. 3sec. $= (537.183)_{12}$: and we have merely to extract the square root of the latter form by the ordinary method:

$$\begin{array}{rl} & \dot{5}3\dot{7}.\dot{1}8\dot{3}0\,(23.76 \\ & 4 \\ 43) & \overline{137} \\ & 109 \\ 467) & \overline{\;2t18} \\ & \;27t1 \\ 4726) & \overline{\;\;23730} \\ & \;\;23730 \\ & \overline{\quad\quad} \end{array}$$

whence, $(23.76)_{12} = 27$ feet 7 inches 6 parts, is the square root.

For practice in this part of the subject, the student is referred to the examples belonging to articles (199) and (200), of the author's *Arithmetic*.

373. With respect to the advantages and disadvantages of the various scales of notation which originate by assigning different values to r, it may be remarked that it would be desirable, in point of practical convenience, to select one wherein the number of figures expressing any given numerical magnitude might be confined within limits not very widely extended. This would further prevent excessive prolixity in the execution of the arithmetical operations: and, by practice, it soon becomes equally easy to perform these operations in any scale, provided its radix be not a very large number.

In article (90), it has been proved that all terminating decimals in the common scale are comprehended in the form $\frac{a}{2^p 5^q}$: so in the senary scale, for instance, all such quantities would be comprised in the form $\frac{a}{2^p 3^q}$, because 2 and 3 are the only prime divisors of 6: but, within given limits, there are evidently more multiples of 3 and its powers than there are of 5 and its powers: and therefore the senary scale would seem to possess an advantage over the denary, at least in the expression of fractional quantities. Similar observations will be applicable to the duodenary scale of notation.

The selection of the scale in common use was therefore probably not made from a comparison of its merits with those of other systems, but from some accidental circumstance, which is now generally supposed to have been that the computation among mankind was first conducted by means of the *Fingers* of both hands, and hence the name of *Digits* has been given to the figures in common use.

This subject will be resumed in the first Appendix: and for a short account of the Notation of the *Greeks* and *Hebrews*, the student is referred to the article *Notation*, in BARLOW's *Mathematical and Philosophical Dictionary*.

CHAPTER XV.

FORMS AND KINDS OF NUMBERS.

FORMS OF NUMBERS.

374. DEF. GENERAL Forms of Numbers are certain Algebraical Formulæ, which, by assigning successive values to one or more of the letters contained in them, are capable of producing in order, all numbers whatever.

375. *If* M *represent any integer, then may every whole number, however small or great, be expressed by one or other of the terms of the series:*

$$Mm, \quad Mm+1, \quad Mm+2, \quad Mm+3, \text{ \&c.}, \quad Mm+(M-1),$$

by assigning a proper value to m.

For, every whole number whatever must either be exactly divisible by M, or must leave for a remainder one or other of the numbers,

$$1, \ 2, \ 3, \text{ \&c.}, \ (M-1):$$

and therefore if a proper value be given to m, it manifestly follows that every whole number will be comprised in the series above mentioned.

The number M which characterises any particular set of forms, is termed its *Modulus*, and its magnitude may be assumed at pleasure.

376. COR. 1. If we give to M, the values 1, 2, 3, &c., in succession, we shall have the following corresponding formulæ:

Modulus.	*Forms of Numbers.*
1,	m :
2,	$2m$, $2m+1$:
3,	$3m$, $3m+1$, $3m+2$:
4,	$4m$, $4m+1$, $4m+2$, $4m+3$:
&c.	&c. :

and in each of these sets, if m be made equal to 0, 1, 2, 3, &c. in order, we shall obtain all numbers in succession.

Thus, to the modulus 4,

if $m=0$, we get 0, 1, 2, 3 :
if $m=1$, ... 4, 5, 6, 7 :
if $m=2$, ... 8, 9, 10, 11 : &c. :

similarly of the other forms: and it is to be observed, that the number of different forms belonging to any modulus, will always be equal to the number of units in that modulus.

377. COR. 2. Hence, if we wish to express any given number n by means of any given modulus M, we have only to divide the former by the latter, and to note the quotient m and the remainder R: for then we shall manifestly have

$$n = Mm + R.$$

Ex. Represent 257 by means of the moduli 6, 11 and 13.

Here, 6) 257 — 42 5, 11) 257 — 23 4, 13) 257 — 19 10:

whence, we have $257 = 6.42 + 5 = 11.23 + 4 = 13.19 + 10$.

378. COR. 3. The number of forms, belonging to any given modulus, may be exhibited in an abbreviated shape by the change of an algebraical sign.

Thus, to the modulus 3, we have the three forms,

$$3m, \quad 3m+1, \quad 3m+2:$$

but since $3m+2 = 3(m+1) - 1 = 3m' - 1$, if $m' = m+1$, it is manifest that all numbers are comprised in the forms,

$$3m \text{ and } 3m \pm 1.$$

Again, to the modulus 5, we have the five forms:

$$5m, \quad 5m+1, \quad 5m+2, \quad 5m+3, \quad 5m+4,$$

which are likewise comprehended in the forms:

$$5m, \quad 5m \pm 1, \quad \text{and} \quad 5m \pm 2.$$

And generally, to the modulus M, the forms will become

$$Mm, \quad Mm \pm 1, \quad Mm \pm 2, \quad Mm \pm 3, \quad \&c.$$

A little consideration will make it appear from this, that any M consecutive whole numbers may be represented by consecutive terms of the series Mm, $Mm+1$, $Mm+2$, &c., provided they be taken in proper order, and corresponding values be given to m.

379. Before we proceed further, we will illustrate the use of these forms in the demonstration of a few Arithmetical Theorems.

(1) The product of any two consecutive numbers is even, or divisible by $1 \,.\, 2$.

For, to the modulus 2, any two consecutive numbers may be expressed by $2m$ and $2m \pm 1$:

$$\therefore \text{ their product } = 2m(2m \pm 1) = 2(2m^2 \pm m),$$

which is divisible by 2, and is therefore an even number: that is, $\dfrac{n(n \pm 1)}{1 \,.\, 2}$ is always integral.

Hence also, the continued product of any collection of consecutive numbers is even.

(2) The product of any two odd numbers is odd, and that of any two even numbers is even.

For, to the modulus 2, any two odd numbers n and n' may be expressed by $2m \pm 1$ and $2m' \pm 1$:

$$\begin{aligned} \therefore\ nn' &= (2m \pm 1)(2m' \pm 1) \\ &= 4mm' \pm 2(m + m') + 1 \\ &= 2\{2mm' \pm (m + m')\} + 1: \end{aligned}$$

which is of the form $2m+1$, and therefore odd: again, to the same modulus, any two even numbers n and n' may be represented by $2m$ and $2m'$:

$$\therefore\ nn' = 2m \times 2m' = 2(2mm'):$$

which is of the form $2m$, and therefore even.

Hence, the continued product of any number of odd numbers is odd, and that of any number of even numbers is even: and the continued product of any number of odd and even numbers together is even.

Also, any power whatever of an odd number is odd, and of an even number even: and the converse.

(3) The product of any three consecutive whole numbers is divisible by $1.2.3$ or 6.

Let n_0, n_1, n_2, n_3, n_4, &c. represent any consecutive whole numbers, each of which will be found in the forms $3m$, $3m+1$, $3m+2$, &c.: and first, let

$$n_1 = 3m,\ \therefore\ n_2 = 3m+1 \text{ and } n_3 = 3m+2:$$
$$\therefore\ n_1 n_2 n_3 = 3m(3m+1)(3m+2):$$

of which $(3m+1)(3m+2)$ being divisible by 1.2, from (1), it follows that $n_1 n_2 n_3$ is divisible by $1.2.3$:

next, let

$$n_1 = 3m+1,\ n_2 = 3m+2 \text{ and } n_3 = 3m+3 = 3(m+1):$$
$$\therefore\ n_1 n_2 n_3 = (3m+1)(3m+2)3(m+1):$$

which is divisible by $1.2.3$, as before:

again,

let $n_1 = 3m+2$, $n_2 = 3m+3 = 3(m+1)$ and $n_3 = 3m+4$:

$$\text{but } n_3 = 3m+4 = 3m+1+3 = n_0+3:$$
$$\begin{aligned}\therefore\ n_1 n_2 n_3 &= n_0 n_1 n_2 + 3n_1 n_2 \\ &= n_0 n_1 3(m+1) + 3n_1 n_2 \\ &= 3\{(m+1)n_0 n_1 + n_1 n_2\}:\end{aligned}$$

and each of the quantities within the brackets being divisible by 1.2, it follows that $n_1 n_2 n_3$ is divisible by $1.2.3$.

Hence, if n be any whole number whatever, expressions of the form $\frac{n(n \pm 1)(n \pm 2)}{1 . 2 . 3}$ will be integral.

(4) The product of any four consecutive whole numbers is divisible by $1 . 2 . 3 . 4$ or 24.

Taking n_0, n_1, n_2, n_3, n_4, n_5, &c. to represent any consecutive whole numbers, each of which will be found in the series $4m$, $4m + 1$, $4m + 2$, $4m + 3$, &c.:

first, let $n_1 = 4m$, $n_2 = 4m + 1$, $n_3 = 4m + 2$, $n_4 = 4m + 3$:

$$\therefore\ n_1 n_2 n_3 n_4 = 4m(4m + 1)(4m + 2)(4m + 3),$$

of which $(4m + 1)(4m + 2)(4m + 3)$ is divisible by $1 . 2 . 3$, from (3):

whence, $n_1 n_2 n_3 n_4$ is divisible by $1 . 2 . 3 . 4$:

next, let $n_1 = 4m + 1$, $n_2 = 4m + 2$, $n_3 = 4m + 3$, $n_4 = 4m + 4$:

$$\therefore\ n_1 n_2 n_3 n_4 = (4m + 1)(4m + 2)(4m + 3)\,4(m + 1),$$

which is manifestly divisible by $1 . 2 . 3 . 4$, as before:

again, let $n_1 = 4m + 2$, $n_2 = 4m + 3$, $n_3 = 4m + 4$, $n_4 = 4m + 5$:

$$\text{now, } n_4 = 4m + 5 = 4m + 1 + 4 = n_0 + 4:$$

$$\begin{aligned}\therefore\ n_1 n_2 n_3 n_4 &= n_0 n_1 n_2 n_3 + 4 n_1 n_2 n_3 \\ &= n_0 n_1 n_2 \, 4(m + 1) + 4 n_1 n_2 n_3 \\ &= 4\left\{(m + 1) n_0 n_1 n_2 + n_1 n_2 n_3\right\}:\end{aligned}$$

and because each of the quantities within the brackets is divisible by $1 . 2 . 3$, it follows that $n_1 n_2 n_3 n_4$ is divisible by $1 . 2 . 3 . 4$:

lastly, let $n_1 = 4m + 3$, $n_2 = 4m + 4$, $n_3 = 4m + 5$, $n_4 = 4m + 6$:

$$\text{now, } n_1 = 4m + 3 = 4m + 7 - 4 = n_5 - 4:$$

$$\begin{aligned}\therefore\ n_1 n_2 n_3 n_4 &= n_2 n_3 n_4 n_5 - 4 n_2 n_3 n_4 \\ &= n_3 n_4 n_5 \, 4(m + 1) - 4 n_2 n_3 n_4 \\ &= 4\left\{(m + 1) n_3 n_4 n_5 - n_2 n_3 n_4\right\},\end{aligned}$$

which is manifestly divisible by $1 . 2 . 3 . 4$, as before.

The same method of demonstration may be applied, whatever be the number of factors employed.

From this it appears, that if n be any whole number whatever, all expressions of the forms $\frac{n(n\pm 1)(n\pm 2)(n\pm 3)}{1.2.3.4}$ are equivalent to integral quantities.

(5) The product of any r consecutive integers is divisible by $1.2.3.$ &c. r.

For, suppose the product of $r-1$ consecutive integers to be divisible by $1.2.3.$ &c. $(r-1)$: then, since any r consecutive whole numbers may be represented by rm, $rm+1$, $rm+2$, &c., $rm+r-1$, and the product of $r-1$ of them $(rm+1)(rm+2)$ &c. $\{rm+r-1\}$ has been supposed divisible by $r-1$: it follows that the product of all the r numbers, or, $rm(rm+1)(rm+2)$ &c. $\{rm+r-1\}$ will be divisible by $1.2.3.$ &c. r: that is, if the theorem, assumed above, be true for any one value of r, it will be true for the next superior value: now, in the preceding subdivisions, it has been proved true, when the values of r are 2, 3, 4:

$\therefore$ it is true, when $r=5$: $\therefore$ when $r=6$, and so on:

and therefore it is universally true that the product of any r consecutive integers is divisible by $1.2.3.$ &c. r.

Hence, if n be any whole number whatever, the expression $\frac{n(n-1)(n-2) \text{ \&c. } (n-r)}{1.2.3. \text{ \&c. } (r+1)}$ will be integral.

(6) If n be any odd number, the product $(n^2+3)(n^2+7)$ will be divisible by 32.

For, let $n=2m+1$: therefore $(n^2+3)(n^2+7)$

$=(4m^2+4m+4)(4m^2+4m+8)=16(m^2+m+1)(m^2+m+2)$:

and since $m^2+m+2=m(m+1)+2$ is an even number by (1), it is manifest that $(n^2+3)(n^2+7)$ is a multiple of 32.

(7) Every square number is of one of the forms, $5m$ or $5m\pm 1$.

For, every number n is of one of the forms, $5m$, $5m \pm 1$, $5m \pm 2$:

$$\therefore\ n^2 = (5m)^2 = 25m^2 = 5(5m^2),\text{ is of the form } 5m:$$

$$n^2 = (5m \pm 1)^2 = 25m^2 \pm 10m + 1 = 5(5m^2 \pm 2m) + 1,$$

which is of the form $5m + 1$:

$$n^2 = (5m \pm 2)^2 = 25m^2 \pm 20m + 4 = 5(5m^2 \pm 4m + 1) - 1,$$

which is of the form $5m - 1$.

(8) The difference of the squares of any two odd numbers is divisible by 8.

For, if $n = 2m + 1$ and $n' = 2m' + 1$,

we have
$$\begin{aligned} n^2 - n'^2 &= (2m + 1)^2 - (2m' + 1)^2 \\ &= 4(m^2 - m'^2 + m - m') \\ &= 4\{m(m + 1) - m'(m' + 1)\}: \end{aligned}$$

whereof the quantity within the brackets being divisible by 2, as appears by (1), it follows that $n^2 - n'^2$ is divisible by 8.

(9) The difference between any cube number and its root, is divisible by 6.

For, $n^3 - n = n(n^2 - 1) = (n - 1)n(n + 1)$, which by (3), is divisible by $1.2.3$ or 6.

Hence, every whole number and its cube, when divided by 6, leave the same remainder.

(10) If n be any whole number not divisible by 3, and p be any whole number whatever, then will either $n^p + 1$ or $n^p + 2$ be divisible by 3.

For, since n is of the form $3m \pm 1$, we have

$$n^p = (3m)^p \pm p(3m)^{p-1} + \frac{p(p-1)}{1 \,.\, 2}(3m)^{p-2} \pm \&c. \pm 1,$$

the last term for the lower sign, being negative or positive according as p is odd or even: whence, in all cases, either $n^p + 1$ or $n^p + 2$ will be a multiple of 3.

(11) On the same principles it is easily proved that all cube numbers to the modulus 4, are of the forms $4m$ and $4m \pm 1$: to the modulus 7, of the forms $7m$ and $7m \pm 1$, and to the modulus 9, of the forms $9m$ and $9m \pm 1$: also, that every fourth power is of one of the forms $5m$ and $5m + 1$: and that every fifth power terminates with the same digit as its root, or that the fifth powers of all numbers, with respect to the modulus 10, are of the same forms, as the numbers themselves.

380. If any number n be divisible by the numbers 2, 3, 5, 7, &c. which are all prime to each other, p, q, r, &c. times respectively in succession, it is manifest that

$$n = 2^p\, 3^q\, 5^r \text{ \&c.}:$$

and if 2, 3, 5, &c. be represented by the general symbols a, b, c, &c., every whole number may be written in the form

$$n = a^p\, b^q\, c^r \text{ \&c.}$$

381. When a number is represented in the form $n = a^p b^q c^r$ &c. it will have $(p + 1)(q + 1)(r + 1)$ &c. different divisors.

For, n is divisible by every term of each of the series $1, a, a^2$, &c., a^p; $1, b, b^2$, &c., b^q; $1, c, c^2$, &c., c^r: &c., which are $p + 1$, $q + 1$, $r + 1$, &c. respectively in number: therefore n is divisible by every term of their continued product, which is

$$\left.\begin{array}{l} 1 + a \;\; + a^2 \;\; + \text{\&c.} + a^p \\ + b + ab + a^2b + \text{\&c.} + a^pb \\ + b^2 + ab^2 + a^2b^2 + \text{\&c.} + a^pb^2 \\ + \text{\&c.} \\ + b^q + ab^q + a^2b^q + \text{\&c.} + a^pb^q \end{array}\right\} \times (1 + c + c^2 + \text{\&c.} + c^r) \text{ \&c.}:$$

now there are $(p + 1)$ terms in $1 + a + a^2 +$ &c. $+ a^p$: and also $(p + 1)$ terms in each of the other lines, which are $q + 1$ in number:

therefore there will be $(p + 1)(q + 1)$ divisors arising from the multiplication of the first two factors,

$$1 + a + a^2 + \text{\&c.} + a^p, \text{ and } 1 + b + b^2 + \text{\&c.} + b^q:$$

also, there are $r+1$ terms in $1+c+c^2+\&c. + c^r$, each of which can be combined with the $(p+1)(q+1)$ divisors above found, and consequently there will be $(p+1)(q+1)(r+1)$ divisors in all: and so on.

The number of divisors expressed by $(p+1)(q+1)(r+1)$ &c. include 1, or the element, and also $a^p b^q c^r$ &c., or the number itself: and they are all different from each other.

Ex. To find the number of divisors of 252.

Here, $252 = 2.126 = 2.2.63 = 2.2.3.21 = 2.2.3.3.7 = 2^2 3^2 7$: whence, the number of divisors, 1 and itself being considered two of them, will be

$$(2+1)(2+1)(1+1) = 18.$$

382. Cor. 1. Conversely, to find a number having a given number of divisors, resolve it into the factors x, y, z, &c.: then, if we assume

$$xyz \ \&c. = (p+1)(q+1)(r+1) \ \&c.:$$

$$\text{or, } p = x-1, \quad q = y-1, \quad r = z-1, \ \&c.:$$

the number required will be expressed by

$$a^p b^q c^r \ \&c. \text{ or, } a^{x-1} b^{y-1} c^{z-1} \ \&c.:$$

and it will be the least when a, b, c, &c. are equal to 2, 3, 5, &c., and the indices are taken in the order of their magnitudes, beginning with the highest.

Ex. Required the general form of a number, having 15 divisors.

Here, $15 = 3.5 = xy$: whence, the number will be expressed generally by $a^2 b^4$: and if $a = 2$ and $b = 3$, the number 324 has exactly 15 divisors, the unit and itself being included.

383. Cor. 2. Hence, we may find a multiplier which will render any number a complete m^{th} power.

For, if $n = a^p b^q c^r$ &c. denote any proposed number, let the required multiplier be $k = a^x b^y c^z$ &c. so that

$$kn = a^{p+x} b^{q+y} c^{r+z} \ \&c.:$$

may be a perfect m^{th} power: then it is evident that each of the indices $p+x$, $q+y$, $r+z$, &c. must either be equal to m, or to some multiple of it, as αm, βm, γm, &c.: whence, we have

$$x = \alpha m - p, \quad y = \beta m - q, \quad z = \gamma m - r, \quad \text{\&c.}:$$

and therefore $k = a^{\alpha m - p} b^{\beta m - q} c^{\gamma m - r}$ &c.: and the least numbers which will answer the purpose, must manifestly correspond to the least values of α, β, γ, &c. which render the indices positive.

Ex. Find a multiplier which will produce a cube number from 63.

Here, $63 = 3.21 = 3.3.7 = 3^2.7$: whence the required multiplier will be $3.7^2 = 147$: and the corresponding cube number is $3^3.7^3 = 21^3 = 9261$.

384. To find the sum of all the divisors of n, when it is capable of being expressed by $a^p b^q c^r$ &c.

Each term of the series $1, a, a^2$, &c., a^p is a divisor of n, and the sum of these $= \dfrac{a^{p+1}-1}{a-1}$: also, each term of the series whose sums are $\dfrac{b^{q+1}-1}{b-1}$, $\dfrac{c^{r+1}-1}{c-1}$, &c. will be divisors of n: moreover, each term of the continued product,

$$(1+a+a^2+\text{\&c.}+a^p)(1+b+b^2+\text{\&c.}+b^q)(1+c+c^2+\text{\&c.}+c^r)\text{\&c.},$$

which includes all these, is a divisor of n: and therefore the sum of all the divisors of n, or of $a^p b^q c^r$ &c., will be equivalent to

$$\left(\frac{a^{p+1}-1}{a-1}\right)\left(\frac{b^{q+1}-1}{b-1}\right)\left(\frac{c^{r+1}-1}{c-1}\right)\text{\&c.},$$

the unit or 1, and the number itself or $a^p b^q c^r$ &c. being both included.

Ex. Required the sum of the divisors of 28, the number itself being excluded.

Here, $28 = 2.14 = 2.2.7 = 2^2.7^1$:

and therefore the sum of its divisors will be

$$\left(\frac{2^3-1}{2-1}\right)\left(\frac{7^2-1}{7-1}\right) = \frac{7 \times 48}{6} = 56:$$

and if the number 28 itself be rejected, the required sum

$$= 56 - 28 = 28.$$

From this it appears, that 28 is equal to the sum of all its aliquot parts $= 1 + 2 + 4 + 7 + 14$: and it is therefore said to be a *perfect number:* also, the numbers 6 and 496 possess the same property.

385. Cor. 1. The number of divisors of any number is odd or even, according as it is a square or not.

If $n = a^p b^q c^r$ &c., the number of different divisors will be

$$(p+1)(q+1)(r+1) \text{ \&c.}:$$

but if n be a square number, the indices p, q, r, &c. are all even numbers: and therefore the product $(p+1)(q+1)(r+1)$ &c. must necessarily be odd;

when n is not a square number, one at least of the indices p, q, r, &c. must be odd, and therefore the product $(p+1)$ $(q+1)(r+1)$ &c. will, in this case, be even.

The converse is also true.

386. Cor. 2. Hence, the number of different ways in which $n = a^p b^q c^r$ &c. can be resolved into two factors, will be

$$\tfrac{1}{2}(p+1)(q+1)(r+1) \text{ \&c.}:$$

because every divisor has another corresponding to it, such that their product $= n$: and if n be not a square number, this will be an integer, since the product is then even: should n however be a square number, and therefore the product be odd, the number of different ways will be

$$\tfrac{1}{2}\left\{(p+1)(q+1)(r+1) \text{ \&c.} + 1\right\},$$

because then two factors are equal, and have been reckoned as only one.

387. COR. 3. If the number of different ways in which n may be resolved into two factors prime to each other, be required, it is evident that if m denote the number of the quantities a, b, c, &c., we have only to make

$$p = q = r = \&c. = 1:$$

and the required number will be

$$\tfrac{1}{2}\{(1+1)(1+1)(1+1) \text{ \&c. to } m \text{ factors}\} = 2^{m-1}.$$

PRIME NUMBERS.

388. DEF. *Prime Numbers* are those which have no divisors except unity and themselves, and therefore cannot be divided into any number of equal integral parts greater than unity; and they are thus distinguished from numbers that are *composite*.

Thus, 2, 3, 5, 7, 11, 13, 17, 19, &c. are prime numbers.

389. *Every prime number greater than* 2, *is of one of the forms* 4m ± 1.

For, to the modulus 4, every number may be expressed by

$$4m, \quad 4m \pm 1, \quad \text{or} \quad 4m \pm 2,$$

whereof the first and last being divisible by 2, cannot contain any prime number greater than 2: and consequently every prime number must be comprised in one of the remaining forms

$$4m \pm 1.$$

Ex. Prime numbers of the form $4m + 1$ are

1, 5, 13, 17, &c.:

and of the form $4m - 1$ are 3, 7, 11, 19, &c.

390. COR. Precisely in the same manner, to the modulus 6, it may be proved that every prime number greater than 3 is of one of the forms $6m \pm 1$: and similar formulæ, but not of equal simplicity, may be deduced when other moduli are adopted.

391. *No algebraical formula whatever can express prime numbers only.*

For, since by article (375), every number may be expressed by the formula

$$n = Mx + R,$$

wherein M may be any whole number whatever; if we suppose that when $x = m$, there is obtained the prime number n_1, so that

$$n_1 = Mm + R,$$

then, when $x = m + rn_1$, we should, taking n_r to denote the corresponding value of n, have

$$n_r = M(m + rn_1) + R = Mm + Mrn_1 + R$$
$$= n_1 + Mrn_1 = (1 + Mr) n_1,$$

which is a composite number: hence, since x may always be such an algebraical expression as to represent any whole number whatever, and such a value may always be given to it as renders n a composite number, it follows that there can exist no formula which contains prime numbers exclusively.

Though no algebraical formula can give prime numbers only, certain formulæ have been discovered, which contain a great many.

Thus, if x be taken equal to any term of the series of natural numbers 0, 1, 2, 3, &c., $x^2 + x + 17$ gives 17 primes in succession: also, $2x^2 + 29$ gives 29, and $x^2 + x + 41$ gives 40 primes, without the intervention of a composite number.

Again, $2^x + 1$ gives 31 primes, by making x equal to the terms of 1, 2, 2^2, 2^3, &c. in order.

392. *The number of prime numbers is indefinitely great.*

For, if possible, let there be a limited nnmber of primes n_1, n_2, &c., n_r, whereof n_r is the greatest; then, it is evident that their continued product

$$n_1 n_2 \text{ \&c. } n_r$$

is divisible by each of them: and consequently that

$$n_1 n_2 \text{ \&c. } n_r + 1$$

is not divisible by any one of them: wherefore, this number must either be a prime number itself, or be divisible by one which is greater than n_r; therefore, in neither case, is n_r the greatest prime number; or, in other words, both the number and magnitudes of prime numbers are indefinitely great.

393. *To determine whether any proposed number is a prime or not.*

If a number n be not a prime, it is evident that we may have

$$n = ab:$$

now, if $b = a$, then $n = a^2$ and $\sqrt{n} = a$, or n is divisible by $\sqrt{n}$:

again, if $b < a$, it is obvious that b is $< \sqrt{n}$, and therefore n is divisible by a quantity less than $\sqrt{n}$:

also, if $b > a$, it is equally manifest that a is $< \sqrt{n}$; whence, as before, n is divisible by a quantity less than $\sqrt{n}$: and it evidently follows that these conclusions will not hold good, unless the number can be resolved into two or more factors: in other words, we have obtained a criterion which will enable us to ascertain whether a number is prime, which is the circumstance of its not being capable of division, by any number either equal to or less than its square root.

394. *If* A *represent any number whatever, and* a, b, c, &c. *denote all the numbers less than* 2A *which are prime to it, then will every prime number greater than the prime factors of* A *be comprised in one or other of the forms*

$$4\mathrm{Am} \pm \mathrm{a}, \quad 4\mathrm{Am} \pm \mathrm{b}, \quad 4\mathrm{Am} \pm \mathrm{c}, \quad \&\mathrm{c}.$$

For, any number when divided by $4A$, must necessarily leave for a remainder one or other of the quantities

$$0, \ \pm 1, \ \pm 2, \ \pm 3, \ \&\text{c.}, \ 2A, \ \text{as appears from (378):}$$

whence, omitting all such remainders as are not prime to $4A$, and retaining the rest as a, b, c, &c., we shall manifestly have

all prime numbers greater than the prime factors of A comprised in the forms

$$4Am \pm a, \quad 4Am \pm b, \quad 4Am \pm c, \quad \&c.$$

Ex. If $A = 1$, then $a = 1$, $b = 0$, &c.: and all prime numbers are contained in the forms $4m \pm 1$;

if $A = 2$, then $a = 1$, $b = 3$, and $c = 0$, &c.:

whence, all prime numbers, greater than 2 are comprehended in the forms $8m \pm 1$ and $8m \pm 3$: and so on.

395. *If* m *be a prime number, the coefficient of every term of the expansion of* $(1 + v)^m$, *except the first and last, is divisible by* m.

For, the coefficient of the r^{th} term of the expansion has been proved in article (248) to be

$$\frac{m(m-1)(m-2)\ \&c.\ (m-r+2)}{1 \cdot 2 \cdot 3 \cdot \&c.\ (r-1)}:$$

and it has been shewn in article (252), that all the coefficients are whole numbers when the index is such; therefore, since m is not divisible by any of the factors of the denominator, it follows that

$$\frac{(m-1)(m-2)\ \&c.\ (m-r+2)}{1 \cdot 2 \cdot 3 \cdot \&c.\ (r-1)}$$

must, of itself, be a whole number; and consequently the coefficient of every term, except the first and last, which do not involve m, must be divisible by m without a remainder: but the same conclusions do not follow when m is composite.

POLYGONAL NUMBERS.

396. Def. *Polygonal Numbers* are the sums of any numbers of terms of certain arithmetical series, in each of which the first term is unity: and they are distinguished into orders dependent upon the common difference.

397. If the common differences of the arithmetical series be 0, 1, 2, 3, 4, &c., we shall have, by means of the expression,

$$s = \left\{2a + (n-1)d\right\}\frac{n}{2},$$

the general terms of the corresponding orders of polygonal numbers equal to

$$n, \quad \frac{n^2+n}{2}, \quad \frac{2n^2+0n}{2}, \quad \frac{3n^2-n}{2}, \quad \frac{4n^2-2n}{2}, \quad \text{\&c.}:$$

and the numbers themselves will be found by giving to n the values, 1, 2, 3, &c. in succession.

398. COR. 1. Hence, if, for the sake of uniformity of system, we designate a series of units by the name of the first order, we shall have the following list of polygonal numbers, in the orders to which they belong:

1.	*Units*,....................	1,	1,	1,	1,	1,	1,	&c.:
2.	*Lineal Numbers*,........	1,	2,	3,	4,	5,	6,	&c.:
3.	*Triagonal Numbers*,...	1,	3,	6,	10,	15,	21,	&c.:
4.	*Quadragonal Numbers*,	1,	4,	9,	16,	25,	36,	&c.:
5.	*Pentagonal Numbers*,..	1,	5,	12,	22,	35,	51,	&c.:
	&c.							

and in this arrangement, if r be the denomination of the order, the common difference of the corresponding arithmetic series will be $r-2$, and we shall have the n^{th} or general term of the polygonal numbers of the r^{th} order equal to

$$\frac{(r-2)n^2-(r-4)n}{2}:$$

from which the polygonal numbers belonging to all the orders may be derived, by assigning the requisite values to r.

Thus, the *r-gonal* numbers are 1, r, $3r-3$, $6r-8$, $10r-15$, $15r-24$, &c.

399. COR. 2. Numbers thus formed are termed polygonal, from the circumstance of their capability of being repres by the figures whose names they bear, the sides of the pol corresponding to the values of n in the formula above gi

Thus, if a dot be taken to represent each of the units in n, we may arrange these dots, when the values of n are 1, 2, 3, &c., in the following order:

1. *Points*, ... &c.:
2. *Lines*, ... &c.:
3. *Trigons*, ... ∴ ⁘ &c.:
4. *Squares*, ... :: ⁞⁞⁞ &c.:

&c.

which are perhaps fanciful representations from which their names may have been derived, rather than arrangements having any connection with the origin of the numbers themselves.

400. Cor. 3. If p_3 denote any triagonal or triangular number, expressed generally by $\frac{n(n+1)}{2}$: we shall have

$$8p_3 + 1 = 4(n^2 + n) + 1$$
$$= 4n^2 + 4n + 1 = (2n + 1)^2;$$

that is, every triagonal number multiplied by 8 and increased by 1, becomes a quadragonal or square number.

Again, if $r = 6$, we have the n^{th} term in the series of hexagonal numbers

$$= \frac{4n^2 - 2n}{2} = \frac{(2n - 1)\,2n}{2},$$

which is manifestly the $(2n - 1)^{\text{th}}$ term in the series of triagonal numbers: and similarly in other instances.

401. Cor. 4. If the magnitude p_r, of a polygonal number of the r^{th} order be given, its place in that order, or what is usually termed its *Root*, may be found by the solution of the quadratic equation,

$$\frac{(r - 2)\,n^2 - (r - 4)\,n}{2} = p_r,$$

from which is obtained

$$n = \frac{r - 4 + \sqrt{8(r-2)p_r + (r-4)^2}}{2(r-2)}.$$

Ex. Required the place of 51 in the series of pentagonal numbers.

In this instance, the denomination r of the order being 5, we have

$$n = \frac{1 + \sqrt{8 \cdot 3 \cdot 51 + 1}}{2 \cdot 3} = \frac{1 + \sqrt{1225}}{6} = 6;$$

that is, 51 is the *sixth* in the order of pentagonal numbers.

402. *To find the sum of* n *terms of the* r$^{\text{th}}$ *order of polygonal numbers.*

Since, by (398), the general term of the polygonal series

$$= \frac{(r-2)n^2 - (r-4)n}{2} = \frac{(r-2)(n^2-n) + 2n}{2}$$

$$= n(n-1)\left(\frac{r-2}{2}\right) + n;$$

we shall manifestly have the sum of n terms of the said series

$$= \{1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \&c. + (n-1)n\}\left\{\frac{r-2}{2}\right\}$$

$$+ 1 + 2 + 3 + \&c. + n:$$

$$\text{but } 1 \cdot 2 = 1^2 + 1,$$

$$2 \cdot 3 = 2^2 + 2,$$

$$3 \cdot 4 = 3^2 + 3,$$

$$\&c.$$

$$(n-1)n = (n-1)^2 + n - 1:$$

$$\therefore\ 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \&c. + (n-1)n$$

$$= 1^2 + 2^2 + 3^2 + \&c. + (n-1)^2 + 1 + 2 + 3 + \&c. +$$

$$= \frac{(n-1)n(2n-1)}{1 \cdot 2 \cdot 3} + \frac{(n-1)n}{2}, \text{ by (3) of p}$$

$$= \frac{(n-1)n(n+1)}{3}: \text{ also, } 1 + 2 + 3 + \&c. + n = \frac{1}{}$$

wherefore the sum of the polygonal series becomes

$$= \frac{(n-1)\,n\,(n+1)\,(r-2)}{2\,.\,3} + \frac{n\,(n+1)}{2}$$

$$= \frac{n\,(n+1)}{1\,.\,2}\left\{\frac{(n-1)\,(r-2)+3}{3}\right\}.$$

Ex. Let r be taken equal to 2, 3, 4, 5, &c. in succession, and denoting the sums of the corresponding orders by s_2, s_3, s_4, s_5, &c. we obtain

$$s_2 = \frac{n\,(n+1)\,(0n+3)}{1\,.\,2\,.\,3}: \quad s_3 = \frac{n\,(n+1)\,(1n+2)}{1\,.\,2\,.\,3}:$$

$$s_4 = \frac{n\,(n+1)\,(2n+1)}{1\,.\,2\,.\,3}: \quad s_5 = \frac{n\,(n+1)\,(3n+0)}{1\,.\,2\,.\,3}: \text{ \&c.}$$

403. The last article furnishes us with the means of ascertaining the number of *Balls*, *Shot* or *Shells* forming any regular *Pile*.

Whenever a pile of balls is complete, it will manifestly finish with a single ball, the number of horizontal courses being the same as the number of balls in one side of the lowest course: consequently, the number of balls constituting such piles, will be represented by the sums of the series of triangular, square, &c. numbers, whose number of terms is equal to the number contained in each side of its base.

Ex. 1. Find the number of shot in a finished triangular pile, the number in one side of the base being 40.

Generally, for triangular numbers, we have

$$s_3 = \frac{n\,(n+1)\,(n+2)}{1\,.\,2\,.\,3};$$

and in this instance $n = 40$:

$$\therefore \text{ the number of shot in the pile} = \frac{40\,.\,41\,.\,42}{1\,.\,2\,.\,3} = 11480.$$

Ex. 2. Required the number of shells contained in a square pile, whose side consists of 20.

Here, by (402), we have $s_4 = \frac{n(n+1)(2n+1)}{1.2.3}$, which when n is made equal to 20, gives the required number

$$= \frac{20.21.41}{1.2.3} = 2870.$$

404. Cor. 1. To find the number of shot in a broken pile of the kind above described, we have merely to compute the numbers which would be contained in the entire pile, were it finished, and in the part which is wanting; and then to take their difference.

Ex. What number of shot is contained in five courses of an unfinished pentagonal pile, when each side of the lowest course consists of 12?

Since, by (402), $s_5 = \frac{n^2(n+1)}{1.2}$, we have the number in the whole pile $= \frac{12.12.13}{1.2} = 936$: also, the number which would be contained in the part wanting will obviously be found by making $n = 12 - 5 = 7$, and is therefore $= \frac{7.7.8}{1.2} = 196$: whence, the number in the broken pile $= 936 - 196 = 740$.

405. Cor. 2. It is sometimes however the practice to pile balls in horizontal courses forming rectangles, which consequently finish in a single row at the top: and it is manifest that the number of courses will, in such cases, be equal to the number of balls in the breadth of the lowest, whilst the number in the finishing row will exceed by 1 the difference of the numbers in the length and breadth of the base. The formulæ, above referred to, will not enable us to determine the numbers of balls in such piles, but we may readily deduce one which will answer that purpose.

Let p and q represent the numbers in the len
breadth of the lowest course, n the number of
upon another: then will

$p-1$, $q-1$; $p-2$, $q-2$; &c., $p-n+1$,

be the numbers in the lengths and breadths of the second, third, &c., n^{th} courses: and the entire number of shot in this pile will be

$$pq+(p-1)(q-1)+(p-2)(q-2)+\&c.+(p-n+1)(q-n+1)$$

$$=pq+pq-(p+q)+1^2+pq-2(p+q)+2^2+\&c.$$

$$+pq-(n-1)(p+q)+(n-1)^2$$

$$=npq-\{1+2+3+\&c.+(n-1)\}(p+q)$$

$$+1^2+2^2+3^2+\&c.+(n-1)^2$$

$$=npq-\frac{(n-1)n}{1.2}(p+q)+\frac{(n-1)n(2n-1)}{1.2.3}$$

$$=\frac{n}{4}\left\{4pq-2(n-1)(p+q)+\frac{2(n-1)(2n-1)}{3}\right\}$$

$$=\frac{n}{4}\left\{(2p-n+1)(2q-n+1)+\frac{(n-1)(n+1)}{3}\right\}:$$

and this enunciated at length is the common practical rule.

If $n=q$, or the pile be a finished one, we shall have the number in the uppermost row $=p-q+1$; and the total number in the pile

$$=\frac{q}{4}\left\{(2p-q+1)(q+1)+\frac{(q-1)(q+1)}{3}\right\}:$$

which, when $q=p$, becomes

$$=\frac{p}{4}\left\{(p+1)^2+\frac{p^2-1}{3}\right\}=\frac{p(p+1)(2p+1)}{1.2.3},$$

the number in a completed square pile, as shewn before.

FIGURATE NUMBERS.

406. Def. *Figurate Numbers* are those, whose n^{th} or general terms are comprised in the expression,

$$\frac{n(n+1)(n+2)(n+3)\,\&c.\,(n+r)}{1.2.3.4.\&c.\,(r+1)}:$$

and they are distinguished into the first, second, third, &c. orders, by assigning to r the values 1, 2, 3, &c. respectively.

407. COR. 1. Hence, corresponding to the values 1, 2, 3, 4, &c. of r, we have, by making n equal to 1, 2, 3, 4, &c. in succession, the following orders of figurate numbers:

First order; $1, 3, 6, 10, \text{\&c.}, \dfrac{n(n+1)}{1.2}$:

Second order; $1, 4, 10, 20, \text{\&c.}, \dfrac{n(n+1)(n+2)}{1.2.3}$:

Third order; $1, 5, 15, 35, \text{\&c.}, \dfrac{n(n+1)(n+2)(n+3)}{1.2.3.4}$:

Fourth order; $1, 6, 21, 56, \text{\&c.}, \dfrac{n(n+1)(n+2)(n+3)(n+4)}{1.2.3.4.5}$:

&c.

and it is obvious that by similar substitutions for n, the r^{th} order will be

$$1, \frac{r+2}{1}, \frac{(r+2)(r+3)}{1.2}, \frac{(r+2)(r+3)(r+4)}{1.2.3}, \text{\&c.},$$

$$\frac{(r+2)(r+3)(r+4)\ \text{\&c.}\ (r+n)}{1.2.3.\ \text{\&c.}\ (n-1)}.$$

408. COR. 2. By the formula from which they are generated, it appears that the general terms of figurate numbers of the first, second, third, &c. orders, are the coefficients of the third, fourth, fifth, &c. terms of the expansions of $(1+x)^{n+1}$, $(1+x)^{n+2}$, $(1+x)^{n+3}$, &c., respectively.

409. *If the* $(n+1)^{th}$ *term of the* r^{th} *order of figurate numbers be multiplied by* n, *the product is equal to* $(r+2)$ *the* n^{th} *term of the* $(r+1)^{th}$ *order.*

th order

$$= n \times \frac{(n+1)(n+2)(n+3) \text{ \&c. } (n+r+1)}{1 . 2 . 3 . 4 . \text{\&c. } (r+1)}$$

$$= (r+2) \times \frac{n(n+1)(n+2)(n+3) \text{ \&c. } (n+r+1)}{1 . 2 . 3 . 4 . \text{ \&c. } (r+2)}$$

$= (r+2)$ times the n^{th} term of the $(r+1)^{\text{th}}$ order.

410. Cor. From this article it appears that the n^{th} term of the $(r+1)^{\text{th}}$ order of figurate numbers $= \dfrac{n}{r+2}$ times the $(n+1)^{\text{th}}$ term of the r^{th} order: and thus the terms of any order may be determined from those of the order which immediately precedes it.

411. *If the* n^{th} *term of the* r^{th} *order of figurate numbers be added to the* $(n+1)^{\text{th}}$ *term of the* $(r-1)^{\text{th}}$ *order, the sum will be the* $(n+1)^{\text{th}}$ *term of the* r^{th} *order.*

For, the n^{th} term of the r^{th} order + the $(n+1)^{\text{th}}$ term of the $(r-1)^{\text{th}}$ order

$$= \frac{n(n+1)(n+2) \text{ \&c. } (n+r)}{1 . 2 . 3 . \text{\&c. } (r+1)} + \frac{(n+1)(n+2)(n+3) \text{ \&c. } (n+r)}{1 . 2 . 3 . \text{\&c. } r}$$

$$= \frac{(n+1)(n+2)(n+3) \text{ \&c. } (n+r)}{1 . 2 . 3 . \text{\&c. } r} \left\{ \frac{n}{r+1} + 1 \right\}$$

$$= \frac{(n+1)(n+2)(n+3) \text{ \&c. } (n+r)(n+r+1)}{1 . 2 . 3 . \text{\&c. } (r+1)},$$

which is the $(n+1)^{\text{th}}$ term of the r^{th} order.

412. Cor. 1. By this proposition is immediately discovered the law of the formation of the terms of any order by means of the terms of the preceding order; for, if to the n^{th} term of any order there be added the $(n+1)^{\text{th}}$ term of the next inferior order, the sum will be the $(n+1)^{\text{th}}$ term of that order: and this will be found to obtain in the orders as stated in (407).

413. Cor. 2. Also, since the first term in every order is 1, it follows that the second term of any order is equal to the sum of the first two terms of the next inferior order: the third term is equal to the first three terms of the preceding order: and generally the n^{th} term of any order is equal to the sum of the first n terms of the order which immediately precedes it. This will readily be observed to be true by reference to the numbers given in (407).

414. *To find the sum of* n *terms of the* r$^{\text{th}}$ *order of figurate numbers.*

By reversing the latter part of the last article, we have the sum of n terms of the r^{th} order equal to the n^{th} term of the $(r+1)^{\text{th}}$ order $= \dfrac{n(n+1)(n+2)\text{ \&c. }(n+r+1)}{1.2.3.\text{\&c.}(r+2)}$.

Ex. If we make r equal to 1, 2, 3, &c., in succession, and denote the corresponding sums by s_1, s_2, s_3, &c., we obtain

$$s_1 = \frac{n(n+1)(n+2)}{1.2.3}:$$

$$s_2 = \frac{n(n+1)(n+2)(n+3)}{1.2.3.4}:$$

$$s_3 = \frac{n(n+1)(n+2)(n+3)(n+4)}{1.2.3.4.5}:$$

&c.

415. Since the figurate numbers of different orders are composed of the sums of series of polygonal numbers, they may be conceived to form pyramids, in the same manner as a series of polygons each less than the other, but of the same number of sides when applied to one another would form a pyramid, and on that account it was formerly usual to term them *Pyramidal* numbers of different orders.

416. The connection subsisting between figurate numbers and the expansions of binomials alluded to in article

(408), induced the earlier mathematicians to pay considerable attention to the laws of their formation, by means of which they obtained the expansion of one power from that which immediately precedes it, the sets of numbers

1, 1, 1, 1, 1, 1, &c.:
1, 2, 3, 4, 5, 6, &c.:
1, 3, 6, 10, 15, 21, &c.:
1, 4, 10, 20, 35, 56, &c.:
1, 5, 15, 35, 70, 126, &c.:
1, 6, 21, 56, 126, 252, &c.:
&c.

whether read horizontally or vertically being termed *Binomial Columns*.

Instead of the law of their generation being deduced from their general forms as was first done by LEGENDRE, the forms of figurate numbers were then determined from the consideration of the manner in which they were produced, which added greatly to the difficulty of the subject; and in this point of view it was treated by FERMAT and others: but the demonstration of the Binomial Theorem in its present shape has now rendered these numbers matters of curiosity rather than of use.

For much important information upon the subjects briefly treated of in this Chapter, the student is referred to the *Essai sur la Theorie des Nombres par* A. M. LEGENDRE, and to BARLOW's *Elementary Investigation of the Theory of Numbers:* also, he will find some additional theorems in the first Appendix of this work.

CHAPTER XVI.

SUMMATION OF SERIES.

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417. DEF. A SERIES is a continued rank, or progression of arithmetical or algebraical quantities connected together by one or both of the signs + and −, and proceeding according to some determinate law.

Thus, we have seen that $1 + 2 + 3 + 4 +$ &c. is an arithmetic series, whose terms increase by the common difference 1: and $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} +$ &c. is a geometric series whose terms are connected throughout by the same common ratio $\frac{1}{2}$.

A *Diverging Series* is one whose terms continually increase, and its value becomes more *divergent* from any quantity that can be assigned, the further it is continued.

Thus, $1 - 2 + 4 - 8 + 16 - 32 +$ &c. is a diverging series.

A *Converging Series* is one whose terms decrease successively, and its value becomes more and more *convergent* towards some quantity which is supposed capable of being assigned.

Thus, $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} +$ &c. is a converging series, for its value has been seen, in example (1) of article (204), to become more and more nearly equal to 2, as the number of terms taken is increased.

418. DEF. *Arithmetical* and *Geometrical* Series, as considered in a preceding chapter, naturally present themselves in the ordinary course of calculations: but series of other forms arise in a variety of ways, and many of them from the expansion

or developement of expressions, according to the generalized processes of symbolical algebra: thus, by algebraical division, we have

$$\frac{1}{a+b} = \frac{1}{a} - \frac{b}{a^2} + \frac{b^2}{a^3} - \frac{b^3}{a^4} + \&c.$$

where the latter member is merely a symbolical expression, equivalent to the former, when no particular values are supposed to be given to the symbols employed: if we suppose $a = b = 1$, this becomes $\frac{1}{2} = 1 - 1 + 1 - 1 + 1 - 1 + \&c.$ which is called a *Neutral Series*, but to which no arithmetical value can be attached, without reference to the mode in which it originated: and this proves its value to be $\frac{1}{2}$.

In the same way, the series $a + ar + ar^2 + \&c.$ to n terms, is an expression equivalent to $\dfrac{a(r^n - 1)}{r - 1}$: and each may at any time be substituted for the other, when convenient, as has been seen in articles (197) and (204).

419. Def. The *Summation of Series* is the expressing the sum of any number of terms by means of a concise general formula: or, it is the determining the nature and form of the equivalent expression from which the series may have been derived by algebraical expansion: thus,

the sum of n terms of $1 + 3 + 5 + 7 + \&c.$ is expressed by n^2: and the sum of $1 + \frac{1}{2} + \frac{1}{4} + \&c.$ *in infinitum* is 2, which when put in the form $\dfrac{2}{2-1}$, is symbolically equivalent to it.

This subject is of far too extensive a nature, to admit of a full discussion in an elementary treatise like the present: and we shall therefore confine our attention to the consideration of a few simple examples, so selected as to point out the different methods that are usually applied, in addition to the formulæ of the eighth chapter.

420. The sums of series may frequently be found by their *Resolution* into two more others, either arithmetical or geometrical.

Ex. 1. Find the sum of n terms of the series

$$1,\ 5,\ 13,\ 29,\ \&c.$$

This $= (4-3) + (8-3) + (16-3) + (32-3) + \&c.$ to n erms,

$$= 4 + 8 + 16 + 32 + \&c. \text{ to } n \text{ terms},$$

$$- \{3 + 3 + 3 + 3 + \&c. \text{ to } n \text{ terms}\}$$

$$= 4 . 2^n - 4 - 3n = 2^{n+2} - 3n - 4.$$

Ex. 2. Required the sum of n terms of the series 1, , 7, 15, 31, &c.

Here, the terms of the series are 1, $(1+2)$, $(1+2+4)$, $1+2+4+8)$, &c.:

nd from the formula $s = \dfrac{a(r^n-1)}{r-1}$, we have the sum

$$= a\frac{r-1}{r-1} + a\frac{r^2-1}{r-1} + a\frac{r^3-1}{r-1} + \&c. \text{ to } n \text{ terms}$$

$$= \frac{a}{r-1}\{(r-1) + (r^2-1) + (r^3-1) + \&c. \text{ to } n \text{ terms}\}$$

$$= \frac{a}{r-1}\{r + r^2 + r^3 + \&c. \text{ to } n \text{ terms} - n\}$$

$$= \frac{a}{r-1}\left\{\frac{r(r^n-1)}{r-1} - n\right\} = \frac{ar(r^n-1)}{(r-1)^2} - \frac{na}{r-1}$$

$$= 2(2^n-1) - n = 2^{n+1} - n - 2, \text{ since } a = 1, \text{ and } r = 2.$$

Ex. 3. Find the value of the expression

$$\frac{1}{\sqrt{2}(1+\sqrt{2})} + \frac{1}{(1+\sqrt{2})(2+\sqrt{2})} + \frac{1}{(2+\sqrt{2})(3+\sqrt{2})} + \&c.$$

n infinitum.

Here, $\dfrac{1}{\sqrt{2}} = \dfrac{1+\sqrt{2}}{\sqrt{2}(1+\sqrt{2})} = \dfrac{1}{\sqrt{2}(1+\sqrt{2})} + \dfrac{1}{1+\sqrt{2}}$:

$$\frac{1}{+\sqrt{2}} = \frac{2+\sqrt{2}}{(1+\sqrt{2})(2+\sqrt{2})} = \frac{1}{(1+\sqrt{2})(2+\sqrt{2})} + \frac{1}{2+\sqrt{2}}:$$

$$\frac{1}{2+\sqrt{2}} = \frac{3+\sqrt{2}}{(2+\sqrt{2})(3+\sqrt{2})} = \frac{1}{(2+\sqrt{2})(3+\sqrt{2})} + \frac{1}{3+\sqrt{2}}:$$

$$\&c. \quad = \quad \&c. \quad = \quad \&c.:$$

whence, by adding, and cancelling the terms which are common to both sides of the result, we have

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}(1+\sqrt{2})} + \frac{1}{(1+\sqrt{2})(2+\sqrt{2})} + \frac{1}{(2+\sqrt{2})(3+\sqrt{2})}$$

+ &c. *in infinitum.*

In the same way, an equivalent expression for an infinite number of terms of this series may be obtained, at whatever term it commence.

421. The sums of series may often be obtained by the method of *Indeterminate Coefficients.*

Ex. 1. Find the sum of n terms of the series 1^2, 2^2, 3^2, 4^2, &c.

Assume $1^2 + 2^2 + 3^2 + \&c. + n^2 = An^3 + Bn^2 + Cn$: then, since the values of A, B, C are independent of the value of n, we shall have

$$1^2+2^2+3^2+\&c.+n^2+(n+1)^2 = A(n+1)^3+B(n+1)^2+C(n+1):$$

whence, by subtraction, we obtain

$$n^2 + 2n + 1 = 3An^2 + (3A + 2B)n + A + B + C:$$

and by equating coefficients, the values of A, B, C are found to be $\frac{1}{3}$, $\frac{1}{2}$, $\frac{1}{6}$ respectively:

$$\therefore \text{ the required sum } = \tfrac{1}{3}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{6}n = \frac{n(n+1)(2n+1)}{1.2.3}:$$

which is the result of (3) of article (216), where it is obtained by a more laborious process.

If higher or lower indices of n had been introduced into the assumed expression, their coefficients would have been found equal to 0.

By a similar assumption, the sum of n terms of the series $1^3, 2^3, 3^3, 4^3$, &c. will be found to be

$$\frac{n^2(n+1)^2}{(1.2)^2}; \text{ and } \therefore\ 1^3+2^3+3^3+4^3+\&c. \text{ to } n \text{ terms},$$

$$= (1+2+3+4+\&c. \text{ to } n \text{ terms})^2.$$

Ex. 2. Required the value of n terms of the series

$$1.2+2.3+3.4+4.5+\&c.$$

Let $1.2+2.3+\&c.+n(n+1) = An^3+Bn^2+Cn$:

$$\therefore\ 1.2+2.3+\&c.+n(n+1)+(n+1)(n+2)$$

$$= A(n+1)^3+B(n+1)^2+C(n+1):$$

whence, by subtraction, we shall have

$$n^2+3n+2 = 3An^2+(3A+2B)n+A+B+C:$$

$$\therefore\ A=\frac{1}{3},\quad B=1, \text{ and } C=\frac{2}{3}:$$

and the sum of the series is $\frac{1}{3}n^3+n^2+\frac{2}{3}n$

$$= \tfrac{1}{3}n(n^2+3n+2) = \tfrac{1}{3}n(n+1)(n+2).$$

This result might have been obtained from the last example: for $1.2 = 1^2+1$, $2.3 = 2^2+2$, $3.4 = 3^2+3$, &c.: and therefore the sum of $1.2+2.3+3.4+\&c.$ to n terms = the sum of $1^2+2^2+3^2+\&c.$ to n terms + the sum of $1+2+3+\&c.$ to n terms

$$= \frac{n(n+1)(2n+1)}{1.2.3}+\frac{n(n+1)}{1.2} = \frac{n(n+1)(n+2)}{3}.$$

Ex. 3. Find the sum of n terms of the series

$$1^2+4^2+7^2+\&c.$$

Here, 1, 4, 7, &c. being in arithmetical progression, the n^{th} term will be $3n-2$: whence, assuming

$$1^2+4^2+7^2+\&c.+(3n-2)^2 = An^3+Bn^2+Cn:$$

we have, $1^2+4^2+7^2+\&c.+(3n-2)^2+(3n+1)^2$

$$= A(n+1)^3+B(n+1)^2+C(n+1):$$

whence, $9n^2 + 6n + 1 = 3An^2 + (3A + 2B)n + A + B + C$:

$$\therefore\ A = 3,\ B = -\tfrac{3}{2} \text{ and } C = -\tfrac{1}{2},$$

so that the general form expressive of the sum of n terms of the series is $\frac{1}{2}n(6n^2 - 3n - 1)$.

The same course may be pursued whatever be the value of the index, provided the quantities be in arithmetical progression: and when n is very large, it will easily be seen that an approximate value of the sum of n terms of the series

$$a^m + (a + d)^m + (a + 2d)^m + \&c.$$

$$\text{is } \frac{n^{m+1}d^m}{m+1}.$$

Ex. 4. Required a general expression for the sum of n terms of the series $1.2.3 + 2.3.4 + 3.4.5 + \&c.$

Let $1.2.3 + \&c. + n(n+1)(n+2) = An^4 + Bn^3 + Cn^2 + Dn$:

$$\therefore\ 1.2.3 + \&c. + (n+1)(n+2)(n+3)$$

$$= A(n+1)^4 + B(n+1)^3 + C(n+1)^2 + D(n+1):$$

whence, by subtraction, we obtain,

$$n^3 + 6n^2 + 11n + 6$$

$$= 4An^3 + (6A + 3B)n^2 + (4A + 3B + 2C)n + A + B + C + D:$$

$$\therefore\ A = \frac{1}{4},\ B = \frac{3}{2},\ C = \frac{11}{4} \text{ and } D = \frac{3}{2}:$$

and the sum required is expressed by

$$\frac{1}{4}n^4 + \frac{3}{2}n^3 + \frac{11}{4}n^2 + \frac{3}{2}n = \frac{n}{4}(n^3 + 6n^2 + 11n + 6)$$

$$= \frac{n(n+1)(n+2)(n+3)}{4}.$$

From this, we may find the sum of $1^3 + 2^3 + 3^3 + \&c.$ to n terms: for since, $n^3 = (n-1)n(n+1) + n$, we have

$$1^3 = \quad 0 \quad + 1:$$
$$2^3 = 1 \cdot 2 \cdot 3 + 2:$$
$$3^3 = 2 \cdot 3 \cdot 4 + 3:$$
$$\&c. = \&c.:$$

$\therefore$ the sum of $1^3 + 2^3 + 3^3 + \&c.$ to n terms

= the sum of $0 + 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \&c.$ to n terms

+ the sum of $1 + 2 + 3 + \&c.$ to n terms

$$= \frac{(n-1)\,n\,(n+1)\,(n+2)}{4} + \frac{n\,(n+1)}{2}$$

$$= \frac{n^2\,(n+1)^2}{(1 \cdot 2)^2}, \text{ as before given.}$$

422. The sums of a great variety of series whose terms are fractional, may be obtained by the method of *Subtraction*.

Ex. 1. Find the sum of n terms of the series

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \&c.$$

$$\text{Let } \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \&c. + \frac{1}{n} + \frac{1}{n+1} = s:$$

$$\therefore \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \&c. + \frac{1}{n+1} = s - 1:$$

$$\text{whence, } \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \&c. + \frac{1}{n\,(n+1)} + \frac{1}{n+1} = 1:$$

which is obtained by subtracting the lower line from the upper:

$$\therefore \text{ the required sum} = 1 - \frac{1}{n+1} = \frac{n}{n+1}.$$

If n be infinite, the sum of the series

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \&c. \textit{ in infinitum} = 1.$$

Ex. 2. Required the sum of n terms of the series

$$\frac{1}{1.3}+\frac{1}{2.4}+\frac{1}{3.5}+\&c.$$

Let $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\&c.+\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}=s$:

$$\therefore\ \frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\&c.+\frac{1}{n+2}=s-\frac{3}{2}:$$

whence, subtracting the latter from the former, we have

$$\frac{2}{1.3}+\frac{2}{2.4}+\frac{2}{3.5}+\&c.+\frac{2}{n(n+2)}+\frac{1}{n+1}+\frac{1}{n+2}=\frac{3}{2}:$$

$\therefore$ the sum of the proposed series will be

$$\frac{1}{2}\left\{\frac{3}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right\}=\frac{3n^2+5n}{4(n+1)(n+2)}=\frac{n(3n+5)}{4(n+1)(n+2)}.$$

When the series is supposed to be indefinitely continued, we have $n=\infty$: and the limit of the result $=\frac{3n^2}{4n^2}=\frac{3}{4}$, so that the sum of the infinite series

$$\frac{1}{1.3}+\frac{1}{2.4}+\frac{1}{3.5}+\&c.\text{ is }\frac{3}{4}.$$

No difficulty will be found in proving that the sum of n terms of the series

$$\frac{1}{a(a+b)}+\frac{1}{(a+b)(a+2b)}+\frac{1}{(a+2b)(a+3b)}+\&c.$$

is $\frac{1}{ab}-\frac{1}{b(a+nb)}$, which, when n is infinite, becomes $\frac{1}{ab}$.

A similar process will shew that the sum of the infinite series $\frac{1}{1.3}-\frac{1}{2.4}+\frac{1}{3.5}-\&c.$ is $\frac{1}{4}$.

Ex. 3. Required the sum of n terms of the series,

$$\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \&c.$$

Let $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \&c. + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} = s$:

$$\therefore \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \&c. + \frac{1}{(n+1)(n+2)} = s - \frac{1}{2}:$$

whence, by subtraction, we have

$$\frac{2}{1.2.3} + \frac{2}{2.3.4} + \&c. + \frac{2}{n(n+1)(n+2)} + \frac{1}{(n+1)(n+2)} = \frac{1}{2}:$$

and the sum of the proposed series will therefore be

$$\frac{1}{2}\left\{\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right\} = \frac{n(n+3)}{4(n+1)(n+2)}.$$

If n be infinite, and therefore 1, 2, 3 be neglected in comparison with it, the sum of the corresponding infinite series will be $\frac{1}{4}$.

On the same principles, it will be easy to prove that the sum of n terms of the series,

$$\frac{1}{a(a+b)(a+2b)} + \frac{1}{(a+b)(a+2b)(a+3b)}$$

$$+ \frac{1}{(a+2b)(a+3b)(a+4b)} + \&c.,$$

is expressed by the compound quantity

$$\frac{1}{2ab(a+b)} - \frac{1}{2b(a+nb)\{a+(n+1)b\}}:$$

which becomes $\frac{1}{2ab(a+b)}$, when n is infinite.

In a manner precisely similar to the one adopted above, may be treated the series

$$\frac{1}{a} + \frac{c}{a(a+b)} + \frac{c+b}{a(a+b)(a+2b)} + \&c.$$

whose sum *in infinitum* will be expressed by $\frac{1}{a-c}$, in which b does not appear.

Also, the sum of the following series,

$$\frac{c+d}{a(a+b)(a+2b)} + \frac{c+2d}{(a+b)(a+2b)(a+3b)} + \&c. \textit{ in infinitum},$$

will be found to be equivalent to $\frac{bc+(a+b)d}{2a(a+b)b^2}$, by assuming

$$\frac{x+y}{a(a+b)} + \frac{x+2y}{(a+b)(a+2b)} + \&c. \textit{ in infinitum} = \sigma:$$

then transposing and subtracting, and equating the numerators to $c+d$, $c+2d$, &c. in succession, from which the values of x and y will be obtained.

423. The sums of a variety of series are easily ascertained by the method of *Multiplication*.

Ex. 1. To find the sum of $\frac{1}{1.3} + \frac{1}{2.4} + \frac{1}{3.5} + \&c.$ *in infinitum*.

Let $1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \&c. \textit{ in infinitum} = \sigma:$

then, multiplying both members by $1-x^2$, we have

$$\left.\begin{aligned} 1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \&c. \\ - x^2 - \frac{x^3}{2} - \&c. \end{aligned}\right\} = (1-x^2)\,\sigma:$$

$$\text{or, } 1 + \frac{x}{2} - \frac{2x^2}{1.3} - \frac{2x^3}{2.4} - \frac{2x^4}{3.5} - \&c. = (1-x^2)\,\sigma:$$

$$\text{whence,}\quad \frac{x^2}{1\cdot3}+\frac{x^3}{2\cdot4}+\frac{x^4}{3\cdot5}+\&c.=\frac{1}{2}+\frac{x}{4}-\frac{1}{2}(1-x^2)\sigma,$$

is universally true, whatever be the value of x:

and in order to eliminate the symbol σ, let $1-x^2=0$, which gives $x=1$, or $x=-1$:

$$\text{therefore,}\quad \frac{1}{1\cdot3}+\frac{1}{2\cdot4}+\frac{1}{3\cdot5}+\&c.\ \textit{in infinitum}=\frac{3}{4}.$$

Also, by using the negative value of x, we have

$$\frac{1}{1\cdot3}-\frac{1}{2\cdot4}+\frac{1}{3\cdot5}-\&c.\ \textit{in infinitum}=\frac{1}{4}.$$

Ex. 2. Required the sum of the infinite series,

$$\frac{5}{1\cdot2\cdot3}+\frac{6}{2\cdot3\cdot4}+\frac{7}{3\cdot4\cdot5}+\&c.$$

$$\text{Assume}\quad 1+\frac{x}{2}+\frac{x^2}{3}+\frac{x^3}{4}+\&c.\ \textit{in infinitum}=\sigma:$$

then, multiplying both sides by $(x-1)(2x-1)$, we have

$$\left.\begin{array}{l} 2x^2+\dfrac{2x^3}{2}+\dfrac{2x^4}{3}+\&c. \\ -3x-\dfrac{3x^2}{2}-\dfrac{3x^3}{3}-\dfrac{3x^4}{4}-\&c. \\ +1+\dfrac{x}{2}+\dfrac{x^2}{3}+\dfrac{x^3}{4}+\dfrac{x^4}{5}+\&c. \end{array}\right\}=(x-1)(2x-1)\sigma:$$

$$\text{whence,}\quad \frac{5x^2}{1\cdot2\cdot3}+\frac{6x^3}{2\cdot3\cdot4}+\frac{7x^4}{3\cdot4\cdot5}+\&c.\ \textit{in infinitum}$$

$$=(x-1)(2x-1)\sigma-1+\frac{5x}{2}:$$

and by making $x=1$, we find the sum required to be $\frac{3}{2}$.

If we take $x=\frac{1}{2}$, the sum of the infinite series,

$$\frac{5}{1.2.3.2^2}+\frac{6}{2.3.4.2^3}+\frac{7}{3.4.5.2^4}+\&c.$$

is found to be $\frac{1}{4}$.

Ex. 3. To find the sum of n terms of the series

$$\frac{c}{a(a+b)}+\frac{c+d}{(a+b)(a+2b)}+\frac{c+2d}{(a+2b)(a+3b)}+\&c.$$

Let $\frac{1}{a}+\frac{x}{a+b}+\frac{x^2}{a+2b}+\&c.+\frac{x^{n-1}}{a+(n-1)b}+\frac{x^n}{a+nb}=s$:

$$\therefore\ (mx-r)s+\frac{r}{a}-\frac{mx^{n+1}}{a+nb}$$

$$=\frac{(m-r)a+mb}{a(a+b)}x+\frac{(m-r)a+(2m-r)b}{(a+b)(a+2b)}x^2+\&c.$$

to n terms:

whence, by assigning proper values to m and r, and assuming $mx-r=0$, or $x=\frac{r}{m}$, the sum of a series of this kind will be determined.

If $m=1$, $r=1$, and therefore $x=1$, we shall have the sum of n terms of the series,

$$\frac{b}{a(a+b)}+\frac{b}{(a+b)(a+2b)}+\&c.=\frac{1}{a}-\frac{1}{a+nb}.$$

If $m=2$, $r=1$, and therefore $x=\frac{1}{2}$, we have

$$\frac{a+2b}{a(a+b)2}+\frac{a+3b}{(a+b)(a+2b)2^2}+\&c.\text{ to } n \text{ terms}$$

$$=\frac{1}{a}-\frac{1}{(a+nb)2^n}.$$

In the applications of this method, it is to be borne in mind, that the multiplier used must always consist of a number of terms equal to that of the consecutive factors in the denominators.

424. The sums of many series may be ascertained by modifications of the *Binomial Theorem*.

Ex. 1. To find the sum of the infinite series,

$$1-\frac{m-1}{2m}x+\frac{(m-1)(2m-1)}{2m\,.\,3m}x^2-\frac{(m-1)(2m-1)(3m-1)}{2m\,.\,3m\,.\,4m}x^3+\&c.$$

$$\text{The sum}=1-\frac{1-\frac{1}{m}}{2}x+\frac{\left(1-\frac{1}{m}\right)\left(2-\frac{1}{m}\right)}{2\,.\,3}x^2-\&c.$$

$$=1+\frac{\frac{1}{m}-1}{2}x+\frac{\left(\frac{1}{m}-1\right)\left(\frac{1}{m}-2\right)}{2\,.\,3}x^2+\&c.$$

$$=\frac{m}{x}\left\{\frac{1}{m}x+\frac{\frac{1}{m}\left(\frac{1}{m}-1\right)}{1\,.\,2}x^2+\frac{\frac{1}{m}\left(\frac{1}{m}-1\right)\left(\frac{1}{m}-2\right)}{1\,.\,2\,.\,3}x^3+\&c.\right\}$$

$$=\frac{m}{x}\left\{(1+x)^{\frac{1}{m}}-1\right\}=\frac{m(1+x)^{\frac{1}{m}}-m}{x}.$$

Ex. 2. Required the value of the infinite series

$$\frac{1}{3}+\frac{3}{2}\cdot\frac{1}{3^2}+\frac{3\,.\,4}{2\,.\,3}\cdot\frac{1}{3^3}+\frac{3\,.\,4\,.\,5}{2\,.\,3\,.\,4}\cdot\frac{1}{3^4}+\&c.$$

From article (265), we have

$$ma^m\left(\frac{x}{a+x}\right)+\frac{m(m+1)}{1\,.\,2}a^m\left(\frac{x}{a+x}\right)^2+\&c.\ \textit{in infinitum}$$

$$=(a+x)^m-a^m:$$

whence, if $ma^m=1$, $m+1=3$, and $\frac{x}{a+x}=\frac{1}{3}$,

we shall find $m=2$, $a=\frac{1}{\sqrt{2}}$, and $x=\frac{1}{2\sqrt{2}}$:

$\therefore$ the required sum $=\left(\frac{3}{2\sqrt{2}}\right)^2-\left(\frac{1}{\sqrt{2}}\right)^2=\frac{5}{8}$.

THE DIFFERENTIAL METHOD.

425. Def. If we have any number of similar quantities, and the differences between the first and second, the second and third, the third and fourth, and so on, be taken, the several remainders form what is called the *First order of Differences:* and if the differences of these differences be taken in the same manner, and the process be continued, the successive results are termed the *second*, *third*, &c., *orders* of differences.

Thus, if the series be the squares of the natural numbers,

1, 4, 9, 16, 25, 36, 49, 64, &c.:

the first order of differences will be

3, 5, 7, 9, 11, 13, 15, &c.:

the second order of differences will be

2, 2, 2, 2, 2, 2, &c.:

and the third order is 0, 0, 0, &c.: that is, there are in this case only two orders of differences, which are finite.

426. If δ_1, δ_2, δ_3, δ_4, δ_5, &c., denote the first terms of the successive orders of differences of the quantities a, b, c, d, e, f, &c.: we shall have

$$\delta_1 = a - b:$$
$$\delta_2 = a - 2b + c:$$
$$\delta_3 = a - 3b + 3c - d:$$
$$\delta_4 = a - 4b + 6c - 4d + e:$$
$$\delta_5 = a - 5b + 10c - 10d + 5e - f: \text{ \&c.}$$

where the numerical coefficients are evidently those of the expansion of $(1 - v)^m$, corresponding to the values 1, 2, 3, 4, 5, &c. of m: and consequently we shall have

$$\delta_r = a - rb + \frac{r(r-1)}{1 \cdot 2} c - \frac{r(r-1)(r-2)}{1 \cdot 2 \cdot 3} d + \text{\&c.}$$

427. From the last article, we have immediately

$$b = a - \delta_1:$$

$$c = -a + 2b + \delta_2 = a - 2\delta_1 + \delta_2:$$

$$d = a - 3b + 3c - \delta_3 = a - 3\delta_1 + 3\delta_2 - \delta_3:$$

$$e = -a + 4b - 6c + 4d + \delta_4 = a - 4\delta_1 + 6\delta_2 - 4\delta_3 + \delta_4:$$

and so on, where the numerical coefficients of δ are those of the expansion of $(1-v)^m$, as before: whence, the n^{th} term of the series will manifestly be

$$a - (n-1)\delta_1 + \frac{(n-1)(n-2)}{1\,.\,2}\delta_2 - \frac{(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3}\delta_3 + \&c.$$

If the terms of the series increase, it will be found in the same manner that the n^{th} term will be expressed by

$$a + (n-1)\delta_1 + \frac{(n-1)(n-2)}{1\,.\,2}\delta_2 + \&c.$$

Ex. Required the n^{th} term of the series of numbers,

$$1, \quad 3, \quad 6, \quad 10, \quad 15, \quad 21, \quad \&c.$$

Here, we have the following orders of differences:

$$2, \quad 3, \quad 4, \quad 5, \quad 6, \quad \&c.:$$

$$1, \quad 1, \quad 1, \quad 1, \quad \&c.:$$

$$0, \quad 0, \quad 0, \quad \&c.:$$

so that $\delta_1 = 2$, $\delta_2 = 1$, $\delta_3 = 0$, &c.:

$$\therefore \text{ the } n^{\text{th}} \text{ term} = 1 + (n-1)\,2 + \frac{(n-1)(n-2)}{1\,.\,2}\,1$$

$$= 1 + 2n - 2 + \frac{n^2 - 3n + 2}{2} = \frac{n^2 + n}{2} = \frac{n(n+1)}{2}.$$

428. Cor. Whenever the differences at last become equal to 0, the value of the n^{th} term will be obtained exactly; but in other cases, the result will be merely an approximation to the true value.

429. *To express the sum of a series, by means of the first terms of the successive orders of differences.*

Since, $a = a$:

$$b = a + \delta_1:$$

$$c = a + 2\delta_1 + \delta_2:$$

$$d = a + 3\delta_1 + 3\delta_2 + \delta_3:$$

$$e = a + 4\delta_1 + 6\delta_2 + 4\delta_3 + \delta_4: \text{ \&c.}$$

we shall have $a + b + c + d + e +$ &c. to n terms

$$= na + \{1 + 2 + 3 + 4 + \text{\&c. to } (n-1) \text{ terms}\}\,\delta_1$$
$$+ \{1 + 3 + 6 + 10 + \text{\&c. to } (n-2) \text{ terms}\}\,\delta_2$$
$$+ \{1 + 4 + 10 + 20 + \text{\&c. to } (n-3) \text{ terms}\}\,\delta_3 + \text{\&c.}$$
$$= na + \frac{n(n-1)}{1.2}\delta_1 + \frac{n(n-1)(n-2)}{1.2.3}\delta_2 + \text{\&c.},$$

as appears from article (414).

Ex. To find the sum of n terms of the series,

$$1^2 + 2^2 + 3^2 + 4^2 + \text{\&c.}$$

Here, we have 1, 4, 9, 16, &c.,

3, 5, 7, &c., or $\delta_1 = 3$:

2, 2, &c., or $\delta_2 = 2$:

0, &c., or $\delta_3 = 0$: &c.

$$\therefore \text{ the sum} = n + \frac{n(n-1)}{1.2}3 + \frac{n(n-1)(n-2)}{1.2.3}2$$

$$= n + \frac{n(n-1)}{1.2}\left\{3 + \frac{2n-4}{3}\right\} = n + \frac{n(n-1)}{1.2.3}(2n+5)$$

$$= \frac{6n + 2n^3 + 3n^2 - 5n}{1.2.3} = \frac{2n^3 + 3n^2 + n}{1.2.3} = \frac{n(n+1)(2n+1)}{1.2.3}.$$

430. The formulæ for the n^{th} term and the sum of n terms of a series, above investigated, are very easily recollected: and whenever any order of differences vanishes, they may

be immediately applied: but the chief use of the method is in the interpolation of series, which will be briefly pointed out in the first Appendix.

THE METHOD OF INCREMENTS.

431. Def. Any quantity susceptible of increase or diminution is called an *Integral*, and the quantity by which it is increased or diminished at any state of its magnitude, is termed its *Increment* or *Decrement*.

Thus, of the arithmetical series 1, 2, 3, &c. n, which is an integral, the sum will be increased by $n+1$, if we proceed one step further: or, the increment of the sum of n terms is here equal to $n+1$.

If s denote the integral, the increment of s, without regard to its algebraical sign, is usually expressed by prefixing the Greek letter Δ before it, and we have in this case $\Delta s = n+1$.

Ex. 1. To find the increment of x^2, having given Δx the increment of x.

$$\text{Here, } \Delta\,(x^2) = (x+\Delta x)^2 - x^2 = 2x\,\Delta x + (\Delta x)^2:$$

$$\text{and if } \Delta x = 1, \text{ we have } \Delta\,(x^2) = 2x+1.$$

Hence also, the integral of $2x+1$ will be x^2+C, where C is a constant quantity to be determined from circumstances, because

$$\Delta\,(x^2+C) = \{(x+1)^2+C\} - x^2 - C = 2x+1,$$

in which C is not found.

The quantity C is termed the *Correction* of the integral, found by reversing the direct process.

Ex. 2. Required the increment of ax^2+bx+c, where the increment of x is 1.

$$\text{Here, } \Delta\,(ax^2+bx+c) = a\,(x+1)^2+b\,(x+1)+c-ax^2-bx-c$$

$$=2ax+a+b.$$

Hence, denoting the reverse operation by the symbol Σ, so that the two processes implied by Δ and Σ neutralize each other, we have

$$\Sigma(2ax + a + b) = \Sigma\Delta(ax^2 + bx + c) = ax^2 + bx + c.$$

432. *To find the increment of the continued product of any number* n *of factors, in arithmetical progression.*

Let s denote the continued product

$$a(a+b)(a+2b) \text{ \&c. } \{a + (n-1)b\},$$

which changes its magnitude by the increase of each factor by b:

$$\text{then } \Delta s = (a+b)(a+2b) \text{ \&c. } \{a+(n-1)b\}(a+nb)$$
$$- a(a+b)(a+2b) \text{ \&c. } \{a+(n-1)b\}$$
$$= (a+b)(a+2b) \text{ \&c. } \{a+(n-1)b\}\{a+nb-a\}:$$
$$= nb(a+b)(a+2b) \text{ \&c. } \{a+(n-1)b\}.$$

433. From this we obtain conversely, the integral of a continued product of factors in arithmetical progression.

For, the integral, $\Sigma(a+b)(a+2b) \text{ \&c. } \{a+(n-1)b\}$

$$= \frac{s}{nb} = \frac{a(a+b)(a+2b) \text{ \&c. } \{a+(n-1)b\}}{nb}:$$

and this in words amounts to the following rule:

Prefix the term of the progression next inferior to the lowest of the proposed quantity, and divide by the common difference multiplied by the number of factors thus increased.

Ex. 1. Find the sum of n terms of the series,

$$1.2 + 2.3 + 3.4 + \text{\&c.}$$

Here, the n^{th} term will manifestly be $n(n+1)$:

and the next term, or $\Delta s = (n+1)(n+2)$:

whence, proceeding according to the rule, we have

$$s = \frac{n(n+1)(n+2)}{3} + C:$$

and when $n = 0$, we know that $s = 0$, which gives $0 = 0 + C$: therefore subtracting the terms of the latter equality from those of the former, we have

$$s = \frac{n(n+1)(n+2)}{3}:$$

and in this case it appears that no correction of the integral is necessary.

Ex. 2. Required the sum of n terms of the series,

$$1^2 + 3^2 + 5^2 + \&c.$$

Here, $\Delta s = (2n+1)^2 = 4n^2 + 4n + 1 = 4n(n+1) + 1$:

$$\therefore\ s = \Sigma 4n(n+1) + \Sigma 1 + C$$

$$= \frac{4(n-1)n(n+1)}{3} + n + C:$$

$$\text{also, } 0 = \quad 0 \quad + 0 + C:$$

$$\therefore\ s = \frac{4(n-1)n(n+1)}{3} + n = \frac{n(2n-1)(2n+1)}{3}.$$

Ex. 3. Determine a general expression for the sum of n terms of the series, $1 + 3 + 6 + 10 + \&c.$

Here the n^{th} term $= \frac{n(n+1)}{2}$, by article (427):

$$\therefore\ \Delta s = \frac{(n+1)(n+2)}{2},$$

being the $(n+1)^{\text{th}}$ term of the series:

$$\therefore\ s = \frac{n(n+1)(n+2)}{1 \,.\, 2 \,.\, 3} + C:$$

also, $0 = 0 + C$: as before:

$$\text{whence, } s = \frac{n(n+1)(n+2)}{1 \,.\, 2 \,.\, 3},$$

which may easily be verified for any particular value of n, and agrees with the result of article (402).

In the same manner, when the general term of the series is $n(n+1)(n+2)$ &c. $(n+r)$, we shall have

$$\Delta s = (n+1)(n+2)(n+3) \text{ \&c. } (n+r+1):$$

whence, $s = \dfrac{n(n+1)(n+2) \text{ \&c. } (n+r+1)}{r+2}$:

and in this way, the sum of n terms of the r^{th} order of figurate numbers may be ascertained.

434. *To find the increment of a fraction, whose denominator consists of* n *factors, in arithmetical progression.*

Let s denote the proposed fraction

$$\frac{1}{a(a+b)(a+2b) \text{ \&c. } \{a+(n-1)b\}}:$$

then $\Delta s = \dfrac{1}{(a+b)(a+2b)(a+3b) \text{ \&c. } (a+nb)}$

$$-\frac{1}{a(a+b)(a+2b) \text{ \&c. } \{a+(n-1)b\}}$$

$$= \frac{1}{(a+b)(a+2b) \text{ \&c. } \{a+(n-1)b\}} \left\{\frac{1}{a+nb} - \frac{1}{a}\right\}$$

$$= -\frac{1}{(a+b)(a+2b) \text{ \&c. } \{a+(n-1)b\}} \frac{nb}{a(a+nb)}$$

$$= -\frac{nb}{a(a+b)(a+2b) \text{ \&c. } (a+nb)}.$$

435. Conversely, we shall have from the result above,

$$\frac{1}{a(a+b)(a+2b) \text{ \&c. } (a+nb)} = -\frac{1}{nb}\Delta s:$$

and therefore the integral of any expression of the form,

$$\frac{1}{a(a+b)(a+2b) \text{ \&c. } (a+nb)},$$

will be $-\dfrac{1}{nb}\dfrac{1}{a(a+b)(a+2b) \text{ \&c. } \{a+(n-1)b\}}$,

derived from it according to the following rule:

Reject the last factor of the denominator, and divide the result by the number of factors left, and by the common difference taken negatively: and thus the integral is found, and may be corrected according to the circumstances of the case in which it is used.

Whenever the factors of the denominator do not form a continued arithmetical progression, but can be rendered so by interpolation, the same rule will manifestly be applicable.

Ex. 1. Find the sum of n terms of the series,

$$\frac{1}{m(m+1)} + \frac{1}{(m+1)(m+2)} + \frac{1}{(m+2)(m+3)} + \&c.$$

If s denote the sum required, we have the n^{th} term

$$= \frac{1}{(m+n-1)(m+n)} : \text{ and } \therefore \Delta s = \frac{1}{(m+n)(m+n+1)} :$$

$$\text{whence, } s = -\frac{1}{m+n} + C :$$

$$\text{also, } 0 = -\frac{1}{m} + C : \text{ since, when } n = 0, \ s = 0 :$$

$$\therefore \text{ by subtraction, } s = \frac{1}{m} - \frac{1}{m+n}, \text{ is the sum required} :$$

$$\text{which, when } n \text{ is infinite, gives } \sigma = \frac{1}{m}.$$

Ex. 2. Required the sum of the series,

$$\frac{1}{1.3.5} + \frac{2}{3.5.7} + \frac{3}{5.7.9} + \&c. \text{ to } n \text{ terms.}$$

$$\text{Here, } \Delta s = \frac{n+1}{(2n+1)(2n+3)(2n+5)}$$

$$= \tfrac{1}{2}\frac{2n+2}{(2n+1)(2n+3)(2n+5)} = \tfrac{1}{2}\frac{(2n+5)-3}{(2n+1)(2n+3)(2n+5)}$$

$$= \tfrac{1}{2}\frac{1}{(2n+1)(2n+3)} - \tfrac{3}{2}\frac{1}{(2n+1)(2n+3)(2n+5)},$$

which is in a proper form for the immediate application of the rule:

$$\therefore\ s = -\tfrac{1}{4}\frac{1}{2n+1} + \tfrac{3}{8}\frac{1}{(2n+1)(2n+3)} + C:$$

$$\text{also, } 0 = -\tfrac{1}{4} + \tfrac{1}{8} + C: \text{ or } C = \tfrac{1}{8}:$$

whence, the required sum will be expressed by

$$s = \tfrac{1}{8} - \tfrac{1}{4}\frac{1}{2n+1} + \tfrac{3}{8}\frac{1}{(2n+1)(2n+3)}:$$

and by making n infinite, we have the sum of the series *in infinitum*, or $\sigma = \frac{1}{8}$.

Ex. 3. Required the sum of n terms of the series,

$$\frac{1}{1.2.4} + \frac{3}{2.3.5} + \frac{5}{3.4.6} + \&c.$$

Here, $\Delta s = \dfrac{2n+1}{(n+1)(n+2)(n+4)}$

$$= \frac{(2n+1)(n+3)}{(n+1)(n+2)(n+3)(n+4)} = \frac{\{2(n+1)-1\}\{(n+2)+1\}}{(n+1)(n+2)(n+3)(n+4)}$$

$$= \frac{2}{(n+3)(n+4)} + \frac{2}{(n+2)(n+3)(n+4)}$$

$$- \frac{n+2}{(n+1)(n+2)(n+3)(n+4)} - \frac{1}{(n+1)(n+2)(n+3)(n+4)}$$

$$= \frac{2}{(n+3)(n+4)} + \frac{1}{(n+2)(n+3)(n+4)}$$

$$- \frac{2}{(n+1)(n+2)(n+3)(n+4)}:$$

$$\therefore\ s = C - \frac{2}{n+3} - \frac{1}{2(n+2)(n+3)} + \frac{2}{3(n+1)(n+2)(n+3)}:$$

$$0 = C - \frac{2}{3} - \frac{1}{2.2.3} + \frac{2}{3.1.2.3}, \text{ or } C = \frac{23}{36}:$$

whence, $s = \frac{23}{36} - \frac{2}{n+3} - \frac{1}{2(n+2)(n+3)} + \frac{2}{3(n+1)(n+2)(n+3)}$:

and therefore the sum *in infinitum*, or $\sigma = \frac{23}{36}$.

Ex. 4. Sum the series

$$\frac{2^2}{1.3.4.5} + \frac{3^2}{2.4.5.6} + \frac{4^2}{3.5.6.7} + \text{\&c. to } n \text{ terms.}$$

Here, $\Delta s = \frac{(n+2)^2}{(n+1)(n+3)(n+4)(n+5)}$

$$= \frac{(n+2)(n+1)+(n+2)}{(n+1)(n+3)(n+4)(n+5)} = \frac{n+2}{(n+3)(n+4)(n+5)}$$

$$+ \frac{n+2}{(n+1)(n+3)(n+4)(n+5)} = \frac{n+2}{(n+3)(n+4)(n+5)}$$

$$+ \frac{1}{(n+3)(n+4)(n+5)} + \frac{1}{(n+1)(n+3)(n+4)(n+5)}$$

$$= \frac{1}{(n+4)(n+5)} + \frac{n+2}{(n+1)(n+2)(n+3)(n+4)(n+5)}$$

$$= \frac{1}{(n+4)(n+5)} + \frac{1}{(n+2)(n+3)(n+4)(n+5)}$$

$$+ \frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)}:$$

$$\therefore s = -\frac{1}{n+4} - \frac{1}{3(n+2)(n+3)(n+4)}$$

$$- \frac{1}{4(n+1)(n+2)(n+3)(n+4)} + C:$$

$$0 = -\tfrac{1}{4} - \frac{1}{3.2.3.4} - \frac{1}{4.1.2.3.4} + C, \text{ or } C = \frac{79}{288}:$$

whence, $s = \dfrac{79}{288} - \dfrac{1}{n+4} - \dfrac{1}{3(n+2)(n+3)(n+4)}$

$$- \frac{1}{4(n+1)(n+2)(n+3)(n+4)}:$$

and the sum of the series *in infinitum*, or $\sigma = \dfrac{79}{288}$.

In the last three examples, the chief part of the process has been the reduction of the increments to the form mentioned in the article.

436. *To find the increment of* a^m, *where* a *is an invariable magnitude, and* m *increases in arithmetical progression.*

Let b be the increment or difference of m, then

$$\Delta a^m = a^{m+b} - a^m = (a^b - 1)\,a^m,$$

is the increment required: and if $b = 1$, we shall have

$$\Delta a^m = (a - 1)\,a^m,$$

which is the formula most frequently wanted.

437. Hence also conversely, we shall have the integral of

$$a^m = \frac{1}{a-1} \times \text{the integral of } \Delta a^m = \frac{a^m}{a-1} + C,$$

where the value of C is to be ascertained as in the preceding cases.

Precisely in the same way we shall have

$$\Delta \frac{1}{a^m} = \frac{1}{a^{m+b}} - \frac{1}{a^m} = \frac{1}{a^m}\left(\frac{1}{a^b} - 1\right)$$

$$= -\left(\frac{a^b - 1}{a^b}\right)\frac{1}{a^m} = -\left(\frac{a-1}{a}\right)\frac{1}{a^m}, \text{ when } b = 1:$$

and $\Sigma \dfrac{1}{a^m} = -\left(\dfrac{a}{a-1}\right)\dfrac{1}{a^m} + C$, by reversing the process.

Ex. 1. Find the sum of n terms of the series,

$$1 + 2 + 4 + 8 + \&c.$$

Here, the n^{th} term is 2^{n-1}, and therefore $\Delta s = 2^n$:

whence, $s = 2^n + C$, by the formula above:

also, $0 = 2^0 + C$, since $s = 0$, when $n = 0$:

$\therefore$ the required sum $s = 2^n - 1$.

Ex. 2. Required a general expression for the value of $+ ar + ar^2 + \&c.$ to n terms.

Here, the n^{th} term $= ar^{n-1}$, and therefore $\Delta s = ar^n$:

$$\text{whence, } s = \left(\frac{a}{r-1}\right) r^n + C:$$

$$\text{also, } 0 = \left(\frac{a}{r-1}\right) r^0 + C:$$

$$\therefore s = a\frac{(r^n - 1)}{r-1}, \text{ is the expression required.}$$

Ex. 3. Determine the sum of n terms of the series,

$$1 + \frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} + \&c.$$

Here, the n^{th} term $= \frac{1}{a^{n-1}}$, and therefore $\Delta s = \frac{1}{a^n}$:

$$\text{whence, } s = -\left(\frac{a}{a-1}\right)\frac{1}{a^n} + C:$$

$$\text{also, } 0 = -\left(\frac{a}{a-1}\right)\frac{1}{a^0} + C:$$

$$\therefore s = \left(\frac{a}{a-1}\right)\left(\frac{1}{a^0} - \frac{1}{a^n}\right) = \frac{a(a^n - 1)}{a^n(a-1)} = \frac{a^n - 1}{a^{n-1}(a-1)}.$$

438. It has before been observed that the chief labour equired in the application of the *Inverse Method of Incre-ents* or in the process of *Integration*, is the resolution of he quantity proposed into factors in arithmetical progression

sometimes termed its *Factorials*: and this, when not immediately apparent, may generally be effected by means of indeterminate coefficients.

Ex. Resolve x^4 into its factorials, beginning with x and increasing by the common difference 1.

Assume $x^4 = Ax(x+1)(x+2)(x+3) + Bx(x+1)(x+2)$
$$+ Cx(x+1) + Dx + E:$$

then, effecting the multiplications of the latter member, we have

$$x^4 = Ax^4 + (6A + B)x^3 + (11A + 3B + C)x^2$$
$$+ (6A + 2B + C + D)x + E:$$

and by equating the corresponding coefficients, we find

$$A = 1,\ B = -6,\ C = 7,\ D = -1, \text{ and } E = 0:$$

$$\therefore\ x^4 = x(x+1)(x+2)(x+3) - 6x(x+1)(x+2)$$
$$+ 7x(x+1) - x:$$

to each term of which the general rule of integration is immediately applicable.

439. The very brief sketch of the Method of Increments here given, which will enable the student to effect the summation of a great variety of series, must suffice for the present, until he arrives at the *Calculus of Finite Differences*, of which this is one of the most elementary applications.

RECURRING SERIES.

440. Def. A *Recurring Series* is so constituted that every term is connected with some number of the terms which precede it, by an invariable law, usually dependant upon the operations of addition, subtraction, &c.

Thus, $1 + 2x + 3x^2 + 4x^3 + 5x^4 + \&c.$ is a recurring series, because the third term $= 2x \times$ the second term $- x^2 \times$ the first term: the fourth term $= 2x \times$ the third term $- x^2 \times$ the second term, and so on, according to the same invariable law, the numerical coefficients 2 and -1 of x and x^2, being termed the *Scale of Relation*.

441. *To find the sum of the infinite series* a + bx + cx² + dx³ + &c., *whose scale of relation is* $\alpha + \beta$.

$$\text{Here,}\quad a = a:$$
$$bx = bx:$$
$$cx^2 = \alpha x\,(bx) + \beta x^2\,(a):$$
$$dx^3 = \alpha x\,(cx^2) + \beta x^2\,(bx):$$
$$ex^4 = \alpha x\,(dx^3) + \beta x^2\,(cx^2):\ \&\text{c.}$$

whence, by addition and substitution, we obtain

$$\sigma = a + bx + \alpha x\,(bx + cx^2 + dx^3 + \&\text{c. } \textit{in infinitum})$$
$$+ \beta x^2\,(a + bx + cx^2 + \&\text{c. } \textit{in infinitum})$$
$$= a + bx + \alpha x\,(\sigma - a) + \beta x^2 \sigma:$$

and $\therefore\ \sigma = \dfrac{a + bx - a\alpha x}{1 - \alpha x - \beta x^2}$, is the sum required.

If the sum of the series after n terms when continued *in infinitum*, be found from this formula by the proper substitutions, it is evident that the sum of the first n terms will be obtained by subtraction.

Ex. 1. Find the sum of the series $1 + 2x + 3x^2 + 4x^3 + \&\text{c.}$

Here, $\alpha = 2$ and $\beta = -1$, as is manifest from a few trials:

$$\therefore\ \sigma = \frac{1 + (2-2)\,x}{1 - 2x + x^2} = \frac{1}{(1-x)^2}:$$

also, after n terms, the series is

$$(n+1)\,x^n + (n+2)\,x^{n+1} + (n+3)\,x^{n+2} + \&\text{c.}:$$

whence, substituting $(n+1)\,x^n$ and $(n+2)\,x^{n+1}$ in the places of a and bx respectively, and retaining the values of α and β, we have

$$\sigma' = \frac{(n+1)\,x^n + (n+2)\,x^{n+1} - 2\,(n+1)\,x^{n+1}}{1 - 2x + x^2}$$
$$= \frac{(n+1)\,x^n - nx^{n+1}}{(1-x)^2}:$$

and therefore the sum of n terms of the series,

$$1 + 2x + 3x^2 + 4x^3 + \&c. = \sigma - \sigma' = \frac{1 - (n+1)x^n + nx^{n+1}}{(1-x)^2}.$$

By changing x into $-x$, we find the sum of n terms of the series $1 - 2x + 3x^2 - 4x^3 + \&c.$ to be expressed by

$$\frac{1 \pm (n+1)x^n \pm nx^{n+1}}{(1+x)^2}:$$

where the upper or lower sign is to be used, according as n is odd or even.

Ex. 2. Required the sum of the recurring series $1 + 3x + 5x^2 + 7x^3 + \&c.$, whose scale of relation is of the form $a + \beta$.

First, $5x^2 = 3ax^2 + \beta x^2$, or $5 = 3a + \beta$:

$7x^3 = 5ax^3 + 3\beta x^3$, or $7 = 5a + 3\beta$:

and from these we find $a = 2$, and $\beta = -1$:

$$\therefore \sigma = \frac{1 + 3x - 2x}{1 - 2x + x^2} = \frac{1 + x}{(1-x)^2}:$$

also, the n^{th} term of the series being $(2n-1)x^{n-1}$,

we have $\sigma' = (2n+1)x^n + (2n+3)x^{n+1} + (2n+5)x^{n+2} + \&c.$

$$= \frac{(2n+1)x^n - (2n-1)x^{n+1}}{(1-x)^2}, \text{ by substitution}:$$

whence, the sum of n terms of the series $= \sigma - \sigma'$

$$= \frac{1 + x - (2n+1)x^n + (2n-1)x^{n+1}}{(1-x)^2}.$$

If the sign of x be changed, the sum of the corresponding series will be had by the same formula.

442. *To find the sum of the infinite series* $\mathrm{a} + \mathrm{b}\mathrm{x} + \mathrm{c}\mathrm{x}^2 + \mathrm{d}\mathrm{x}^3 + \mathrm{e}\mathrm{x}^4 + \mathrm{f}\mathrm{x}^5 + \&c.$, *whose scale of relation is* $a + \beta + \gamma$.

Here, $a = a$:

$bx = bx$:

$cx^2 = cx^2$:

$$dx^3 = \alpha x\,(cx^2) + \beta x^2\,(bx) + \gamma x^3\,(a):$$

$$ex^4 = \alpha x\,(dx^3) + \beta x^2\,(cx^2) + \gamma x^3\,(bx):$$

$$fx^5 = \alpha x\,(ex^4) + \beta x^2\,(dx^3) + \gamma x^3\,(cx^2): \text{\&c.}$$

whence, $\sigma = a + bx + cx^2 + \alpha x\,(cx^2 + dx^3 + ex^4 + \text{\&c.})$

$+ \beta x^2\,(bx + cx^2 + dx^3 + \text{\&c.}) + \gamma x^3\,(a + bx + cx^2 + \text{\&c.})$

$= a + bx + cx^2 + \alpha x\,(\sigma - a - bx) + \beta x^2\,(\sigma - a) + \gamma x^3 \sigma:$

$$\text{and } \therefore\ \sigma = \frac{a + bx + cx^2 - \alpha x\,(a + bx) - a\beta x^2}{1 - \alpha x - \beta x^2 - \gamma x^3}.$$

By the same mode of proceeding, it will appear that the sum of an infinite recurring series will always be expressed in a similar form, whatever be the number of terms composing the scale of relation: and that the sum of n terms may be obtained by the plan above adopted.

Ex. To find the sum of $1^2 + 2^2x + 3^2x^2 + 4^2x^3 + \text{\&c.}$, whose scale of relation is $\alpha + \beta + \gamma$.

Here, we have $16 = 9\alpha + 4\beta + \gamma:$

$$25 = 16\alpha + 9\beta + 4\gamma:$$

$$36 = 25\alpha + 16\beta + 9\gamma:$$

from which are readily found $\alpha = 3$, $\beta = -3$ and $\gamma = 1$:

$$\therefore\ \sigma = \frac{1 + 4x + 9x^2 - 3x - 12x^2 + 3x^2}{1 - 3x + 3x^2 - x^3} = \frac{1 + x}{(1 - x)^3}:$$

$$\text{also, } \sigma' = \frac{(n+1)^2x^n - (2n^2 + 2n - 1)\,x^{n+1} + n^2x^{n+2}}{(1 - x)^3}:$$

and thus, the sum of n terms of the series will be

$$\frac{1 + x - (n+1)^2x^n + (2n^2 + 2n - 1)\,x^{n+1} - n^2x^{n+2}}{(1 - x)^3}.$$

443. Whenever the series is known to be a recurring one, and the number of terms in the scale of relation is given, there can be no difficulty in finding the sum both *in infinitum* and to n terms: and when the form of the

scale of relation of a recurring series is not known, a few trials will in general enable us to discover it, unless it be very far extended.

444. From what has preceded, it appears that the sum of a recurring series, whose scale of relation is $a+\beta+\gamma+$&c., is always capable of being expressed in the form,

$$\frac{A + Bx + Cx^2 + Dx^3 + \&c.}{1 - ax - \beta x^2 - \gamma x^3 - \&c.}.$$

Hence, conversely, if we assume the more general form,

$$\frac{a + a_1 x + a_2 x^2 + \&c.}{b + b_1 x + b_2 x^2 + \&c.} = A + A_1 x + A_2 x^2 + \&c.,$$

we shall have $a + a_1 x + a_2 x^2 + \&c.$

$$= (A + A_1 x + A_2 x^2 + \&c.)\,(b + b_1 x + b_2 x^2 + \&c.)$$

$$\begin{aligned} = Ab &+ A_1 b x + A_2 b x^2 + \&c. \\ &+ A b_1 x + A_1 b_1 x^2 + A_2 b_1 x^3 + \&c. \\ &\qquad + A\, b_2 x^2 + A_1 b_2 x^3 + A_2 b_2 x^4 + \&c. \\ &\qquad\qquad + \&c.: \end{aligned}$$

and equating the coefficients of the same powers of x,

$$Ab = a:$$

$$A_1 b + A b_1 = a_1:$$

$$A_2 b + A_1 b_1 + A b_2 = a_2: \ \&c.$$

which shews that the coefficients A, A_1, A_2, &c. are connected with each other according to a certain determinate law: also, that the number of these *connecting* equations will be the same as the number of dimensions of x in the denominator, and consequently that the scale of relation will consist of as many terms as the said number of dimensions, because the general term will be of the form,

$$A_n = -\frac{b_1}{b} A_{n-1} - \frac{b_2}{b} A_{n-2} - \frac{b_3}{b} A_{n-3} - \&c.$$

445. *Every recurring series is equivalent to as many geometrical progressions, as there are terms in its scale of relation.*

For, since every recurring series originates from the expansion of the form $\frac{a + a_1 x + a_2 x_2 + \&c.}{b + b_1 x + b_2 x^2 + \&c.}$, which may generally be resolved into as many fractions with binomial denominators of the form $p + qx$ as there are units in the highest power of x in the denominator, and each of these may by actual division be made to assume the form of a geometrical progression, it follows that the number of geometrical progressions must be the same as that of the terms in the scale of relation.

446. Whenever these geometrical progressions can be determined *a priori*, the n^{th} term and the sum of n terms of the recurring series will be obtained at once by means of them: but this process is little less laborious, and involves higher principles, than what has been pointed out in the preceding pages.

Ex. Resolve $1 - \frac{5}{2}x + \frac{7}{4}x^2 - \frac{9}{8}x^3 + \&c.$ into its constituent geometrical progressions.

Here, the scale of relation is $-\frac{1}{2}, +\frac{1}{2}$:

$$\text{and the sum} = \frac{2 - 4x}{2 + x - x^2} = \frac{2 - 4x}{(2 - x)(1 + x)}$$

$$= \frac{2}{1 + x} - \frac{2}{2 - x} = 2\left(\frac{1}{1 + x}\right) - \left(\frac{1}{1 - \frac{1}{2}x}\right)$$

$$= 2\{1 - x + x^2 - x^3 + \&c. \textit{ in infinitum}\}$$

$$- \{1 + \tfrac{1}{2}x + \tfrac{1}{4}x^2 + \tfrac{1}{8}x^3 + \&c. \textit{ in infinitum}\}:$$

but as we have first found the *generating* expression, we should be enabled to form a better judgment of it, than of its symbolical developements, though they will readily furnish the means of discovering the n^{th} term, and the sum of n terms of the series: thus, the n^{th} term

$$= \left(\pm 2 - \frac{1}{2^{n-1}}\right) x^{n-1},$$

and the sum of n terms

$$= 2\left\{\frac{1 \pm x^n}{1 + x}\right\} - \frac{1}{2^{n-1}}\left\{\frac{x^n - 2^n}{x - 2}\right\},$$

where the higher or lower sign of the former term in each is applicable, according as n is odd or even.

In this chapter, the student will perceive that much of the detail of the particular examples has been omitted, because it is supposed that by this time he has acquired a facility in all the fundamental operations of the science: and for further information connected with its subdivisions, he is referred to *Bonnycastle's* Algebra: *Emerson's* Increments: *Demoivre's* Miscellanea Analytica: *Euler's* Analysis Infinitorum: the Acta Petropolitana, and most of the periodical Scientific Journals.

CHAPTER XVII.

THE APPLICATION OF ALGEBRA TO GEOMETRY.

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447. DEF. IN the preceding pages the numerical values of concrete magnitudes have frequently been represented by algebraical symbols: and it is on the same principle that the objects of *Geometry*, or *Geometrical Magnitudes*, as *Lines* or *Distances*, *Superficies* or *Areas*, and *Solid Contents* or *Volumes*, are valued or compared by means of the *symbols* representing their respective *dimensions:* also, a line, having length only, is considered as possessed of only *one* dimension: a superficies having both length and breadth comprises *two* dimensions, and a solid has *three* dimensions, inasmuch as it is defined by means of three magnitudes, *length*, *breadth*, and *depth* or *thickness*.

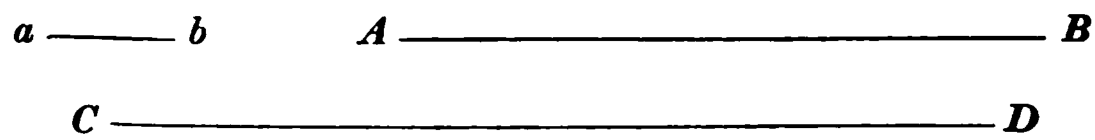
448. DEF. A *Measure* in Geometry is a certain magnitude assumed as an *Unit*, with which other magnitudes of the same kind may be compared: and though one magnitude neither contains another, nor is contained in it, an exact number of times, there may still be a third and smaller magnitude which is capable of *measuring* them both. A *Measure* thus defined has therefore the same relation to *quantity*, as the *unit* or 1 has to *number*; and all quantities and numbers are said to be *equal* to the aggregates or sums of their measures and units respectively.

It appears, therefore, that when the magnitudes of lines are once numerically expressed, the Theorems of Geometry must themselves furnish the means of valuing or comparing with each other, those of both superficies and solids, whereof

lines are the natural dimensions: and on this account we shall first establish the Theory of *Lineal Measure*, and then deduce those of *Superficial* and *Solid Measure* from it.

LINEAL OR LONG MEASURE.

449. Def. An *Unit* of lineal or long measure, is a line of a certain length, arbitrarily fixed upon: and by the determination of the ratios which other lines bear to it, we are enabled to compare with each other, all magnitudes of that description: thus, if the line ab be considered the lineal unit, and be denoted by λ; the numerical magnitude of the straight line AB, will



be determined from the following proportion:

the magnitude of AB : the magnitude of ab :: the lineal units in AB : a lineal unit:

that is, the numerical magnitude of $AB =$ the magnitude of $ab \times \dfrac{\text{the lineal units in } AB}{\text{a lineal unit}} =$ the magnitude of $ab \times$ the *number* of lineal units in AB: and denoting this number, which may be either integral, fractional, or incommensurable, by p, we have the magnitude of AB represented by λp, or by p, when the unit λ is taken identical with the arithmetical unit or 1.

Also, if $\dfrac{\text{the lineal units in } CD}{\text{a lineal unit}}$ be numerically equivalent to q, we have

the magnitude of AB : the magnitude of $CD = p : q$:

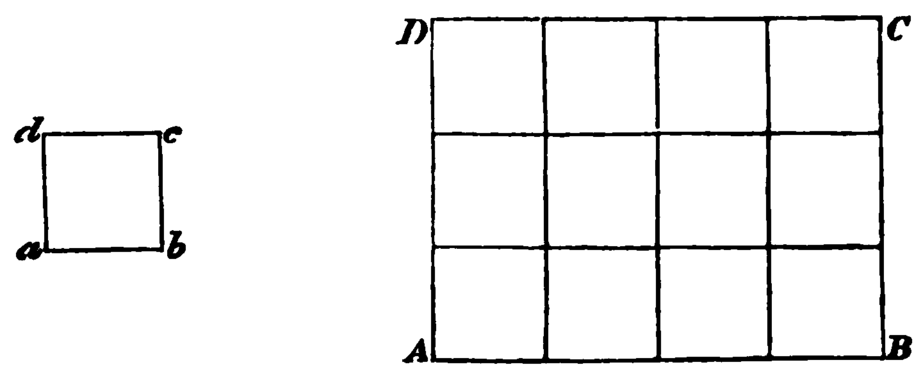
$$\text{whence, } \frac{\text{the magnitude of } AB}{\text{the magnitude of } CD} = \frac{p}{q}:$$

and consequently in the valuations or comparisons of the magnitudes of any lines AB, CD, the algebraical symbols p and q

thus obtained, may always be used instead of the magnitudes themselves: and the algebraical operations when properly applied, will produce symbolical results accordant with the relations of the geometrical magnitudes which they represent.

SUPERFICIAL OR SQUARE MEASURE.

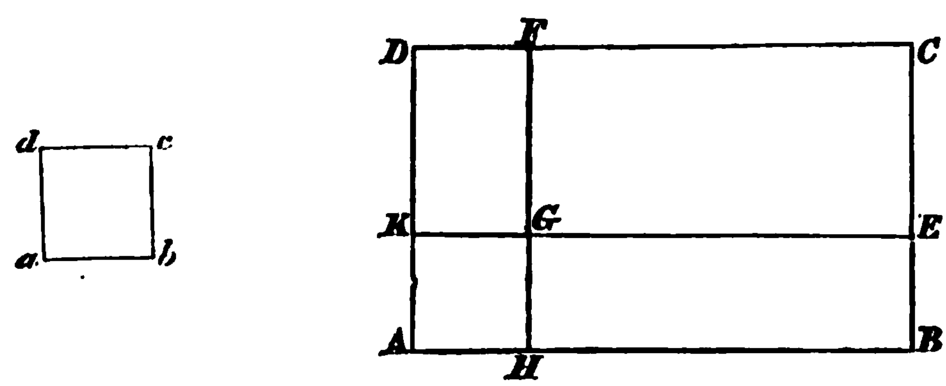
450. Def. An *Unit* of superficial or square measure is a square surface or area, whereof the dimension of each side is equal to the lineal unit: thus, if ab represent the lineal unit,



the square $abcd$ described upon it will be the superficial or square unit, having the two dimensions ab and ad, which may be regarded as its length and breadth: and the arithmetical magnitude of any proposed surface or area, will manifestly be obtained by finding what multiple, part or parts, the surface or area is of this unit.

451. *The numerical value of the area of a rectangular parallelogram, is equal to the product of those of its adjacent sides.*

Let $ABCD$ be a rectangular parallelogram, whereof the adjacent sides AB and AD contain p and q lineal units respectively:



spectively: take $AH = AK =$ the lineal unit, and draw KE and HF parallel to AB and AD intersecting in G, so that the

square AG being equal to the square $abcd$, may represent the superficial unit: then, by *Euclid* VI. 1, we have

$$\text{area of } ABEK : \text{area of } AHGK = AB : AH = p : 1:$$

$$\therefore \text{ area of } ABEK = p \times \text{area of } AHGK:$$

$$\text{also, area of } ABCD : \text{area of } ABEK = AD : AK = q : 1:$$

$$\therefore \text{ area of } ABCD = q \times \text{area of } ABEK$$

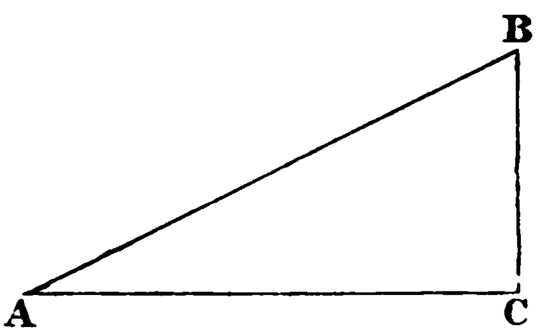
$$= q \times p \times \text{area of } AHGK, \text{ by what precedes:}$$

$$\text{that is, the area of } ABCD = pq \times \text{the area of } AHGK$$

$= pq$, when the magnitude of the superficial unit $AHGK$ is supposed to be identified with the arithmetical unit 1.

If $q = p$, the parallelogram $ABCD$ becomes a square, and we have the area of the square whose side is p, represented algebraically by p^2: that is, if A denote the area of the square whose side is p, we have $A = p^2$, and $\therefore p = \sqrt{A}$, expressing their relations by algebraical symbols.

Ex. 1. Let the base AC and the perpendicular altitude BC of the triangle ABC, right-angled at C, be equal to p and



q lineal units respectively: then, denoting AB by r, we have from *Euclid* I. 47,

$$AB^2 = AC^2 + BC^2: \text{ and } \therefore r^2 = p^2 + q^2:$$

$$\text{whence, the value of } r = \sqrt{p^2 + q^2}, \text{ is obtained.}$$

Also, if $q = p$, we have

$$AB = r = \sqrt{p^2 + p^2} = \sqrt{2p^2} = p\sqrt{2}:$$

that is, if the side AC of a square be a rational numerical magnitude, the diagonal AB whose arithmetical magnitude is

the surd quantity $p\sqrt{2}$, is incommensurable with it. See article (182).

452. By means of the formula $r = \sqrt{p^2 + q^2}$, we may trace the origin of all the primitive arithmetical surds: and conversely their geometrical representations.

Thus, if $p = 1$, and $q = 1$, we have $r = \sqrt{2}$:

$p = \sqrt{2}$, ... $q = 1$, ... $r = \sqrt{3}$:

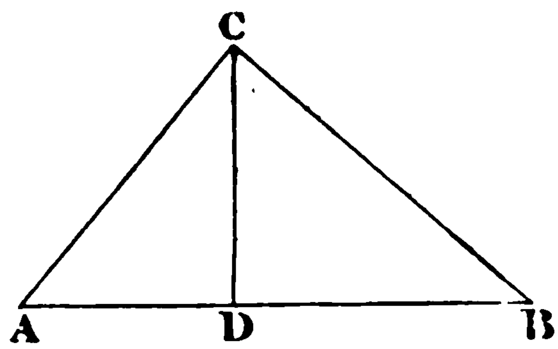
$p = 2$, ... $q = 1$, ... $r = \sqrt{5}$:

$p = 2$, ... $q = \sqrt{2}$, ... $r = \sqrt{6}$: &c.

and all these quantities, though their exact arithmetical values can never be ascertained, are quite distinct and clear when received in connexion with the geometrical magnitudes which they respectively represent.

Ex. 2. Given the base and perpendicular altitude of a triangle, to find its area.

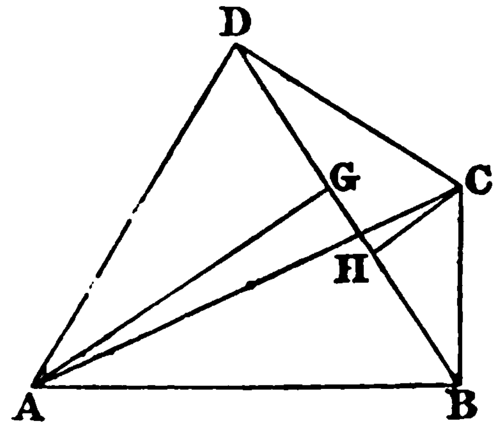
Let the base $AB = p$ and perpendicular altitude $CD = q$, according to the principles of article (449):



then, by *Euclid* I. 41, the area of the triangle ABC is equal to half the rectangular parallelogram whose base is AB and perpendicular altitude $CD = \frac{1}{2} AB \times CD = \frac{1}{2} pq$: that is, if the base and altitude be equivalent to p and q *lineal* units, the area of the triangle will be equivalent to $\frac{1}{2} pq$ *superficial* units of the same denomination.

In the same way, if we have the four-sided figure $ABCD$ called a trapezium,

and find the lineal magnitude of the diagonal BD equivalent



to p, and those of the perpendiculars AG, CH let fall upon it from the angles A, C to q and r respectively, we shall have

$$\begin{aligned}\text{the area of } ABCD &= \text{the area of } ABD + \text{the area of } BCD\\ &= \tfrac{1}{2}BD \times AG + \tfrac{1}{2}BD \times CH\\ &= \tfrac{1}{2}pq + \tfrac{1}{2}pr\\ &= \tfrac{1}{2}p\,(q+r):\end{aligned}$$

that is, the area of a trapezium is found by multiplying half a diagonal, by the sum of the perpendiculars let fall upon it from the opposite angles.

Similarly, the area of any rectilineal figure may be found, by adding together the areas of the triangles of which it is made up.

Also, conversely, the value of any one of the dimensions of the figure here used, may be ascertained, when the rest together with the area, are given.

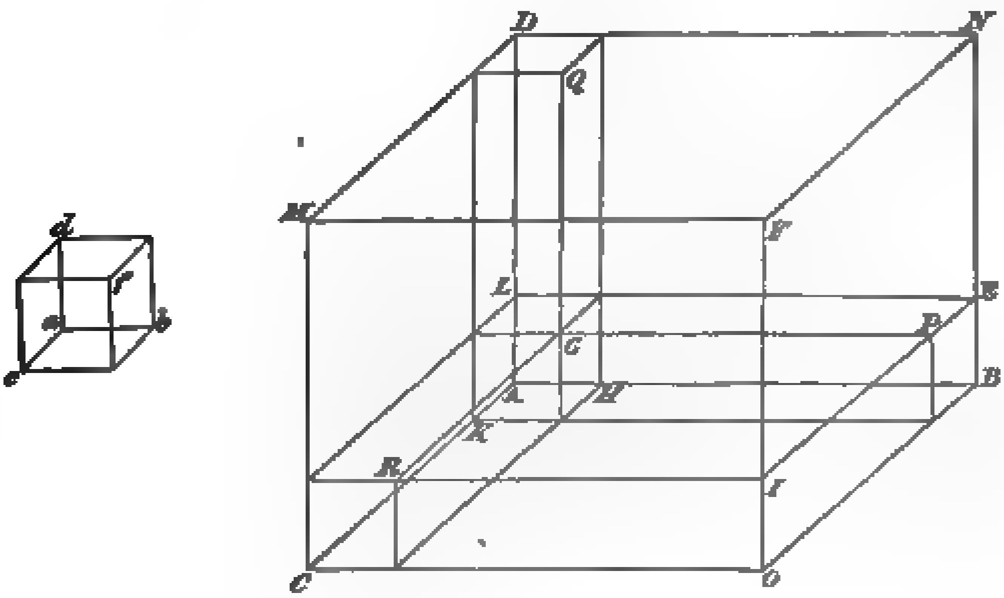
SOLID OR CUBIC MEASURE.

453. DEF. An *Unit* of solid or cubic measure, is a cube or rectangular parallelopiped, whose length, breadth and thickness are each equal in magnitude to the lineal unit: thus, the solid af represented below, wherein $ab = ac = ad =$ the lineal unit as before, denotes the solid unit: and the solid content or volume of any other body of three dimensions will evidently be ascertained by finding what multiple, part or parts, it is of this unit, the lineal dimen-

sions, or the length, breadth and thickness being first determined.

454. *The numerical value of the solid content or volume of a rectangular parallelopiped, is equal to the continued product of those of its length, breadth and thickness.*

Let $ABFM$ represent a rectangular parallelopiped, whereof the length $AB = p$, the breadth $AC = q$, and the thickness $AD = r$, the denominations of the dimensions being the same in each:



take $AH = AK = AL =$ the lineal unit, and complete the construction as in the diagram: then it is manifest that AG will be a cube whose magnitude is equal to that of the solid unit af: and by *Euclid* XI. 25, we have

$$\text{the parallelopiped } AF : \text{the parallelopiped } AI$$
$$= \square\, CD : \square\, CL = AD : AL = r : 1:$$

$$\therefore \text{ the parallelopiped } AF = r \times \text{the parallelopiped } AI:$$

$$\text{also, the parallelopiped } AI : \text{the parallelopiped } AP$$
$$= \square\, BI : \square\, BP = AC : AK = q : 1:$$

$$\therefore \text{ the parallelopiped } AI = q \times \text{the parallelopiped } AP:$$

$$\text{again, the parallelopiped } AP : \text{the parallelopiped } AG$$
$$= \square\, BL : \square\, HL = AB : AH = p : 1:$$

$\therefore$ the parallelopiped $AP = p \times$ the parallelopiped AG:

whence, we have now, the parallelopiped AF

$= r \times$ the parallelopiped AI

$= r \times q \times$ the parallelopiped AP

$= r \times q \times p \times$ the parallelopiped AG

$= pqr$, since the parallelopiped $AG = 1$:

that is, the volume of a rectangular parallelopiped, is expressed by the continued product of its lineal dimensions.

If $AD = AC$, or $r = q$, the volume of the parallelopiped will be pq^2: and if the three edges AB, AC, AD be all equal to each other and to p, the parallelopiped becomes a cube whose volume $= p^3$.

Hence, if V be assumed to represent the volume,

$$\text{we have } V = pqr,\ \ p = \frac{V}{qr},\ \ q = \frac{V}{pr},\ \text{ and }\ r = \frac{V}{pq}:$$

$$\text{also, } V = pq^2,\ \ p = \frac{V}{q^2},\ \text{ and }\ q = \sqrt{\frac{V}{p}}:$$

again, $V = p^3$, and $p = \sqrt[3]{V}$, in the respective cases above considered: and thus, the relations between the corresponding geometrical magnitudes have been expressed algebraically, according to the principles before established.

In the same manner, it appears from the Elements of Geometry that the volume of any parallelopiped whatever is found by multiplying the area of any of its plane faces, by its perpendicular altitude with reference to it.

455. Before we proceed further, we will illustrate the theorems already laid down, by their application to a few useful practical examples.

Ex. 1. A rectangular court, the sides of which are p and q yards, has a path r yards wide cut off from two of its sides: required the area of the path, and of the remaining portion.

Here, the area of the whole court $= pq$:

$$\text{also, the area of the portion left} = (p - r)(q - r)$$

$$= pq - (p + q)r + r^2:$$

$$\therefore \text{ the area of the path} = (p + q)r - r^2.$$

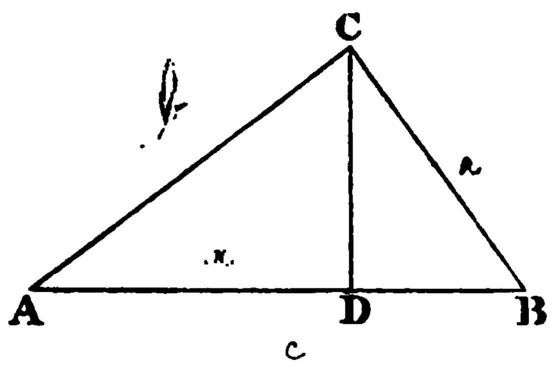
If the path be continued for every side, we shall have the area of the portion left $= (p - 2r)(q - 2r)$

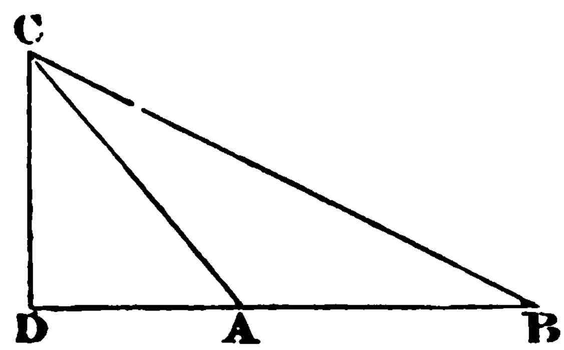
$$= pq - 2(p + q)r + 4r^2:$$

$$\text{and } \therefore \text{ the area of the path} = 2(p + q)r - 4r^2,$$

which differs from twice that found above by $2r^2$, as it manifestly ought to do.

Ex. 2. Given the three sides of a triangle, to find its perpendicular altitude, and the corresponding segments of the base.





Let $AB = c$, $AC = b$, and $BC = a$: also, $x = AD$:

$$\therefore DB = c - x:$$

$$\text{now, } CD^2 = AC^2 - AD^2 = b^2 - x^2:$$

$$\text{also, } CD^2 = CB^2 - DB^2 = a^2 - (c - x)^2:$$

$$\text{whence, } a^2 - c^2 + 2cx - x^2 = b^2 - x^2:$$

$$\therefore x = \frac{b^2 + c^2 - a^2}{2c} = AD:$$

$$\text{and } c - x = \frac{a^2 + c^2 - b^2}{2c} = BD.$$

Hence,

$$CD^2 = b^2 - \left(\frac{b^2 + c^2 - a^2}{2c}\right)^2 = \frac{(2bc)^2 - (b^2 + c^2 - a^2)^2}{4c^2}$$

$$= \frac{(2bc + b^2 + c^2 - a^2)(2bc - b^2 - c^2 + a^2)}{4c^2}$$

$$= \frac{\{(b + c)^2 - a^2\}\{a^2 - (b - c)^2\}}{4c^2}$$

$$= \frac{(a + b + c)(a + b - c)(a + c - b)(b + c - a)}{4c^2},$$

and $CD = \dfrac{1}{2c}\sqrt{(a + b + c)(a + b - c)(a + c - b)(b + c - a)}$:

which, if $2s = a + b + c$ and $\therefore\ 2(s - a) = b + c - a$, $2(s - b) = a + c - b$, and $2(s - c) = a + b - c$, becomes

$$CD = \frac{2}{c}\sqrt{s(s - a)(s - b)(s - c)}.$$

A similar demonstration, and the same results will be found to obtain, when the perpendicular falls without the base of the triangle.

See Propositions 12 and 13 of *Euclid*, Book II.

Ex. 3. To express the area of a triangle in terms of its sides.

Using the diagrams and notation of the last example, we have the area

$$= \tfrac{1}{2} AB \times CD$$

$$= \frac{c}{2} \times \frac{2}{c}\sqrt{s(s - a)(s - b)(s - c)}$$

$$= \sqrt{s(s - a)(s - b)(s - c)},$$

which enunciated at length gives the following rule:

From half the sum of the sides, subtract each side separately: multiply the half sum and the three remainders together, and the square root of the product will be the area.

Ex. 4. Required the area of a triangle whose sides are $\sqrt{2a^2+2b^2-c^2}$, $\sqrt{2a^2+2c^2-b^2}$, and $\sqrt{2b^2+2c^2-a^2}$.

By the last example, (the area)^2

$$= \tfrac{1}{4} AB^2 \times CD^2$$

$$= \frac{c'^2}{4} \times \frac{\{(b'+c')^2 - a'^2\}\{a'^2 - (b'-c')^2\}}{4c'^2}$$

$$= \frac{1}{16}\{(b'+c')^2 - a'^2\}\{a'^2 - (b'-c')^2\}$$

$$= \frac{9}{16}(a+b+c)(a+b-c)(a+c-b)(b+c-a), \text{ by substitution:}$$

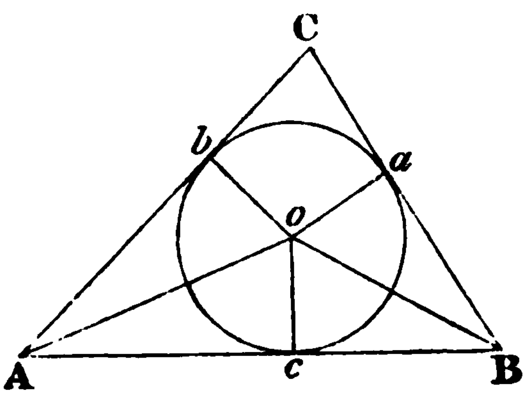
therefore the area required

$$= \tfrac{3}{4}\sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)}$$

= 3 times the area of a triangle whose sides are a, b, c.

Ex. 5. To express the radii of the inscribed and circumscribed circles of a plane triangle, in terms of the sides.

Let o be the centre of the circle inscribed in the triangle



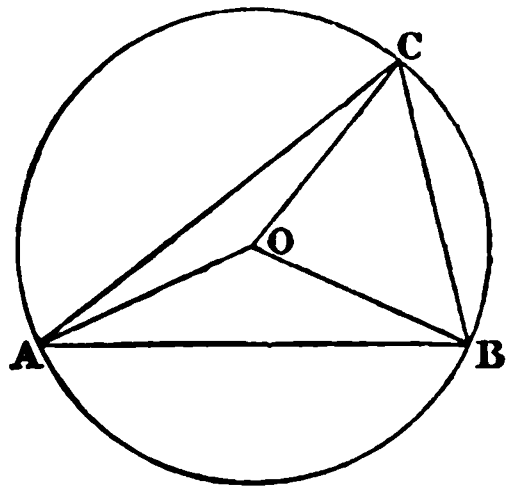
ABC, so that $oa = ob = oc =$ the radius r: then, if S denote the area of the triangle, we have

$$S = \Delta AoB + \Delta AoC + \Delta BoC$$

$$= \tfrac{1}{2}cr + \tfrac{1}{2}br + \tfrac{1}{2}ar = \tfrac{1}{2}(a+b+c)r = rs:$$

$$\text{whence, } r = \frac{S}{s} = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}.$$

Next, let O be the centre of the circle circumscribed about the triangle ABC, so that $OA = OB = OC =$ the radius R: then, if CD be drawn perpendicular



to AB, we have by *Euclid* VI. C, $2R \times CD = AC \times BC$:

$$\text{whence, } R = \frac{ab}{2CD} = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}.$$

From this it appears that $2Rr = \frac{2S}{s}\,\frac{abc}{4S} = \frac{abc}{a+b+c}$.

Ex. 6. To find the volumes of a triangular prism and of a pyramid with a triangular base.

If a rectangular parallelopiped be cut by a diagonal plane, so as to divide it into two parts exactly similar and equal to each other, it is evident that two of its opposite and parallel surfaces are likewise bisected; whence the solid content of each of the upright prisms with right-angled triangles for their bases, into which the parallelopiped is divided, will be obtained by multiplying the quantity expressing the area of its base by that denoting the height or edge at right angles to it.

Also, since by *Euclid* XII. 7, any prism having a triangular base, may be divided into *three* pyramids that have triangular bases and are equal to one another, it is manifest that the solid content of a triangular prism will be obtained by multiplying the area of its base by *one third* of its perpendicular altitude.

Again, because every other upright prism and pyramid may be divided into prisms and pyramids respectively, with triangular bases, we may conclude generally, that the volumes of all upright prisms and pyramids will be expressed in the same terms.

What has been proved of upright solids, holds good of all solids whatever of the same kind, as appears from *Euclid* XI. 31, and XII. 6.

The following results which are not attainable by the application of Algebra to Geometry without reference to higher principles, may be found in the introductory chapter of the *Author's Principles of the Differential Calculus.*

(1) If r be the radius of a circle, and $\pi = 3.14159$ &c., then will the circumference be expressed by $2\pi r$, and the area by πr^2.

Whence, of different circles the circumferences are proportional to the radii, and the areas to the squares of the radii.

Also, if a be the arc of a circular sector whose radius is r, its area will be $\frac{1}{2}ra$.

(2) If r be the radius of the base of a right cylinder, and h its perpendicular height: then, the convex surface $= 2\pi rh$: the whole surface $= 2\pi r(r+h)$: and the volume $= \pi r^2 h$.

(3) If r be the radius of the base of a right cone, h its perpendicular height, and l the length of its side: then will the convex surface $= \pi rl$: the whole surface $= \pi r(r+l)$: and the volume $= \frac{1}{3}\pi r^2 h$.

If R and r be the radii of the ends of the frustum of a right cone, h its height, and l the length of its side:

then, the convex surface $= \pi(R+r)l$:

the whole surface $= \pi\{R(R+l) + r(r+l)\}$:

and the volume $= \frac{1}{3}\pi h\{R^2 + Rr + r^2\}$.

(4) If r be the radius of a sphere, the convex surface $= 4\pi r^2$: and the volume $= \frac{4}{3}\pi r^3$.

These values of the surfaces and solid contents of what are termed the *Round Bodies*, are of very considerable practical importance: and they are deduced by means of a very simple process in the work above referred to.

456. We have now seen how geometrical magnitudes in general may be expressed by means of the symbols of Algebra, and the converse of the preceding propositions will of course obtain: but algebraical quantities considered with respect to *number* only, may always be represented by lines, according to the following principle.

Let p, q, r, &c. be represented by lines:

$$\text{then, since } p : q :: q : \frac{q^2}{p},$$

a third proportional to the lines p and q will represent

$$\frac{q^2}{p}, \text{ or } q^2 \text{ if } p=1, \text{ or } \frac{1}{p} \text{ if } q=1.$$

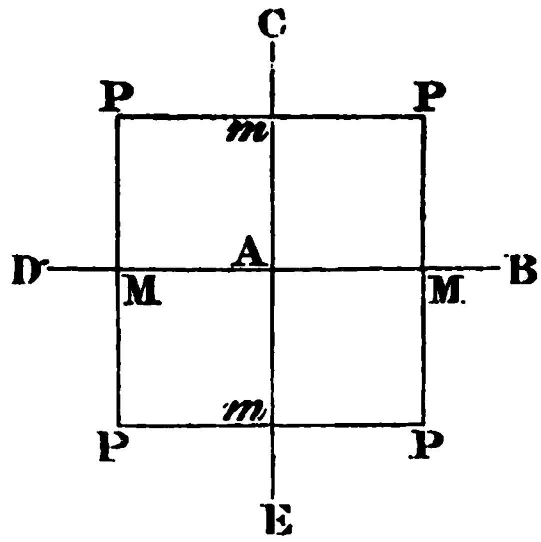
Again, since $p : q :: r : \dfrac{qr}{p}$, a fourth proportional to the lines p, q and r will represent $\dfrac{qr}{p}$, or qr if $p=1$, or q^2 if $r=q$ and $p=1$: and the same may be extended to other quantities thus represented.

Also, if q be a mean proportional between p and r, we have $p : q :: q : r$, or $q^2 = pr$: that is, $q = \sqrt{pr}$, or $\sqrt{pr}$ is represented by a line which is a mean proportional between the lines p and r: and this becomes $\sqrt{p}$, when $r=1$.

The same process is applied to represent by lines such quantities as $\sqrt[4]{pr}$ and $\sqrt[4]{p}$: but for surds whose indices are odd numbers, or involve odd factors, the principles of Geometry, as at present understood will not be sufficient, and approximate values only can be obtained.

457. *To determine the algebraical affections of geometrical magnitudes.*

Let DAB be a straight line, along which, from any fixed



point A in it, different distances are to be measured and estimated; then, if M be any point in AB, we have

$$AM = AB - MB,$$

which, since AB is greater than MB, is evidently a positive quantity: but if M be situated in AD, then MB is greater than AB, and therefore the corresponding value of AM becomes negative, and we have

$$AM = MB - AB = -(AB - MB):$$

whence, if lines measured from the fixed point A towards the *right* be called *positive*, lines measured from the same point A towards the *left* must be termed *negative*. In the same manner, if lines measured from A *upwards* along AC be positive, those measured from A *downwards* along AE will be negative.

The converse is evidently true: and of course the same *affections* belong to all *parallel* lines drawn in the same *direction*.

Also, if the algebraical value of a line be a *negative* quantity, the line must be measured in a direction *opposite* to that which is assumed to be positive.

458. The following examples will furnish illustrations of the use of the preceding propositions.

Ex. 1. Given the difference between the side and the diagonal of a square, to find its area.

Draw a square $ABCD$, whose diagonal is AC, and let the given difference $= d$: then, if $AB = x$, we have

$$AC = \sqrt{x^2 + x^2} = x\sqrt{2}:$$

whence, $x\sqrt{2} - x = d$, by the question:

$$\therefore\ x = \frac{d}{\sqrt{2} - 1} = (1 + \sqrt{2})\, d, \text{ the side of the square:}$$

and its area $= (1 + \sqrt{2})^2 d^2 = (3 + 2\sqrt{2})\, d^2$.

If we had taken $AC = -x\sqrt{2}$, which is admissible by reason of the radical sign, we should have $-x\sqrt{2} - x = d$, an equality which plainly cannot subsist unless d be affected with the negative sign: and this would entirely alter the nature of the problem.

Ex. 2. Produce a given line, so that the rectangle of the given line and the whole line produced, may be equal to the square of the part produced.

Let AB the given straight line $= a$, BC the produced part $= x$: then, $AC = a + x$:

and $a(a + x) = x^2$, or $x^2 - ax = a^2$, by the question:

whence, $x = \frac{1}{2}a(1 \pm \sqrt{5})$, one value of which is positive and the other negative.

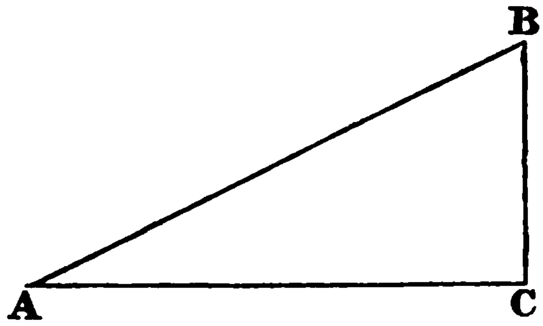
Hence, the value $x = \frac{1}{2}a(1 + \sqrt{5})$ is alone consistent with the *production* of the given line, and answers the condition of the question: but the value

$$x = \tfrac{1}{2}a(1 - \sqrt{5}) = -\tfrac{1}{2}a(\sqrt{5} - 1),$$

accords with the position of C between the points A and B, and therefore gives the solution of Proposition 11 of the second Book of *Euclid's* Elements.

The student will find no difficulty in constructing for the solution of the question proposed: thus, upon AB describe the square $ABDG$: bisect BD in E, join AE and produce DB, so that $EF = EA$: then the side of the square described upon DF, will cut AB produced in the required point C, as is easily demonstrated.

Ex. 3. With a given hypothenuse, find a right-angled triangle whose sides are in continued proportion.



Let $AB = c$, $BC = x$, and $\therefore$ $AC = \sqrt{c^2 - x^2}$:

$$\text{whence, } c : \sqrt{c^2 - x^2} = \sqrt{c^2 - x^2} : x, \text{ by the question:}$$

$$\therefore\ cx = c^2 - x^2, \text{ and } x^2 + cx = c^2,$$

which solved gives $x = \frac{1}{2}c(\sqrt{5} - 1)$:

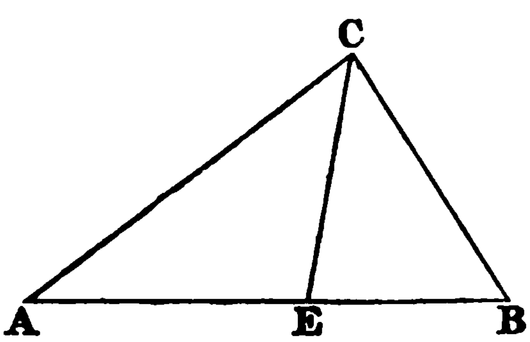
$$\text{and } \therefore\ \sqrt{c^2 - x^2} = \tfrac{1}{2}c\sqrt{2\sqrt{5} - 2}:$$

or the sides are c, $\frac{1}{2}c\sqrt{2\sqrt{5} - 2}$, and $\frac{1}{2}c(\sqrt{5} - 1)$.

The value $x = -\frac{1}{2}c(\sqrt{5} + 1)$ renders $\sqrt{c^2 - x^2}$ imaginary, and therefore furnishes no real solution of the problem.

From this result a geometrical construction may easily be derived.

Ex. 4. Given two sides of a triangle, and the length



of the line bisecting the included angle and meeting the base, to find the triangle.

Let $BC = a$, $AC = b$, $CE = d$, and $AB = x$: then, by Euclid VI. 3, we have

$$AE : EB = AC : BC = b : a:$$

$$\therefore AE : x = b : a + b, \text{ or } AE = \frac{bx}{a+b}:$$

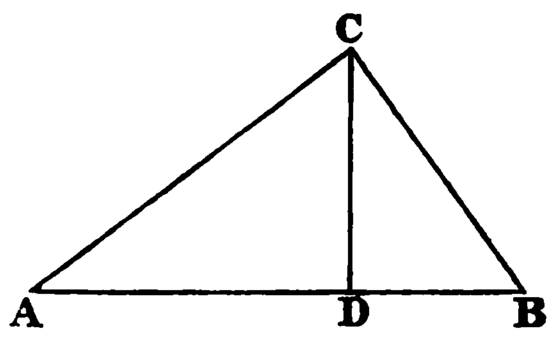
$$\text{and } x : EB = a + b : a, \text{ or } BE = \frac{ax}{a+b}:$$

$$\text{whence, } ab = d^2 + \frac{abx^2}{(a+b)^2}, \text{ by } Euclid \text{ VI. } B:$$

and this solved gives $x = (a+b)\sqrt{\dfrac{ab - d^2}{ab}}$.

From this result it follows that d must be less than $\sqrt{ab}$ to ensure the existence of a triangle with the prescribed data: also, when $d = \sqrt{ab}$, the triangle vanishes; and when d is greater than $\sqrt{ab}$, the data are incongruous.

Ex. 5. Given the segments of the base made by a perpendicular from the opposite angle, and the ratio of the remaining sides of a triangle, to find it.



Let $AD = p$, $DB = q$, and $\dfrac{BC}{AC} = m$: then, if $BC = x$, and $AC = y$, we have $x = my$:

$$\text{also, } x^2 - y^2 = (BD^2 + DC^2) - (AD^2 + DC^2) = q^2 - p^2:$$

$$\text{whence, } x = m\sqrt{\frac{q^2 - p^2}{m^2 - 1}}, \text{ and } y = \sqrt{\frac{q^2 - p^2}{m^2 - 1}},$$

are the values of the remaining sides.

The solution shews that $q^2 - p^2$ and $m^2 - 1$, must have the same algebraical sign in order to ensure the existence of a triangle: and when $m = 1$, we must have $q = p$, so that x and y assume the form $\frac{0}{0}$, in which case the problem becomes indeterminate, the data being insufficient for its solution.

Ex. 6. In a given circle, to inscribe a rectangle of given area.

In the diagram of article (457), suppose a circle to be described with the centre A passing through the points B, P, C, D, E: and let $AB = a$, $AM = x$:

then, by *Euclid* III. 35, we have $DM \cdot MB = MP^2$:

$$\therefore\ MP = \sqrt{(a+x)(a-x)} = \sqrt{a^2 - x^2}:$$

whence, if k^2 be the given area, we shall have

$$4x\sqrt{a^2 - x^2} = k^2, \text{ or } 16x^4 - 16a^2x^2 = -k^4:$$

$$\therefore\ 16x^4 - 16a^2x^2 + 4a^4 = 4a^4 - k^4:$$

$$\therefore\ 4x^2 - 2a^2 = \pm\sqrt{4a^4 - k^4}, \text{ and } x = \pm\tfrac{1}{2}\sqrt{2a^2 \pm \sqrt{4a^4 - k^4}}:$$

$$\text{that is, } AM = \pm\tfrac{1}{2}\sqrt{2a^2 \pm \sqrt{4a^4 - k^4}},$$

$$\text{and } MP = \pm\tfrac{1}{2}\sqrt{2a^2 \mp \sqrt{4a^4 - k^4}}:$$

and the four values of each of these lines admit of their geometrical interpretations, by reference to the four rectangles situated about the centre of the circle.

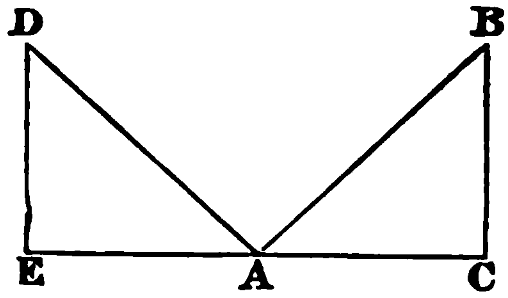
It is obvious also that k^2 cannot be greater than $2a^2$; and when $k^2 = 2a^2$, or is the *greatest possible*, we have

$$AM = \pm\frac{a}{\sqrt{2}}, \text{ and } MP = \pm\frac{a}{\sqrt{2}},$$

so that the rectangle becomes a square whose area $= 2a^2$.

Ex. 7. From two given points draw two straight lines to a given line, so that their sum shall be of a given magnitude.

Draw BC and DE perpendicular to the given line CAE:



also let $BC = a$, $DE = b$, $EC = c$, $AC = x$, and $\therefore AE = c - x$:

then, $\sqrt{a^2 + x^2} + \sqrt{b^2 + (c-x)^2} = AB + AD = d$, suppose:

and this by reduction becomes

$$4(d^2 - c^2)x^2 - 4c(a^2 - b^2 - c^2 + d^2)x$$
$$+ \left\{4a^2d^2 - (a^2 - b^2 - c^2 + d^2)^2\right\} = 0,$$

the solution of which gives the required values of x.

When the two values of x become equal to each other, the former member becomes a complete square, so that four times the product of the extremes is equal to the square of the mean; whence, effecting the multiplications, &c., we find $d^2 = (a+b)^2 + c^2$: and the equation becomes

$$4(a+b)^2x^2 - 8ac(a+b)x + 4a^2c^2 = 0:$$

$$\therefore\ 2(a+b)x - 2ac = 0, \quad \text{or } x = \frac{ac}{a+b}:$$

$$\text{whence, } c - x = \frac{bc}{a+b}, \text{ and therefore } \frac{x}{c-x} = \frac{a}{b}:$$

that is, $\dfrac{AC}{BC} = \dfrac{AE}{DE}$, and the triangles BAC, DAE are similar:

also, it will readily appear that the value of d is then the least possible.

If D be on the other side of the given line, the solution will be the same: and when $d^2 = (a+b)^2 + c^2$, BA and AD form a straight line, which is manifestly the least possible.

Ex. 8. To find the side of a regular decagon inscribed in a circle of given radius.

Taking the diagram of *Euclid* IV. 10, we shall manifestly have BD = the side of the decagon inscribed in the circle whose radius is AB: then, if $BD = AC = x$, and $AB = r$,

$$\therefore\ BC = r - x, \text{ and } r(r - x) = x^2:$$

from which, $x = \frac{1}{2}r(\sqrt{5} - 1)$ the side required.

Ex. 9. To divide the area of a circle into n equal parts, by means of concentric circumferences.

Let r = the radius of the circle: then πr^2 = the whole area, and each part $= \dfrac{\pi r^2}{n}$:

let x_1, x_2, x_3, &c. be the radii of the required circumferences taken in order:

$$\therefore\ \pi x_1^2 = \frac{\pi r^2}{n}, \text{ or } x_1 = r\sqrt{\frac{1}{n}}:$$

$$\pi x_2^2 = \frac{2\pi r^2}{n}, \text{ or } x_2 = r\sqrt{\frac{2}{n}}: \text{ \&c.}:$$

that is, the radii of the circles beginning with the innermost are

$$r\sqrt{\frac{1}{n}}\quad r\sqrt{\frac{2}{n}},\quad r\sqrt{\frac{3}{n}}, \text{ \&c., } r\sqrt{\frac{n-1}{n}}:$$

and the breadths of the concentric annuli will therefore be

$$r\left(\frac{\sqrt{n} - \sqrt{n-1}}{\sqrt{n}}\right),\quad r\left(\frac{\sqrt{n-1} - \sqrt{n-2}}{\sqrt{n}}\right), \text{ \&c.}:$$

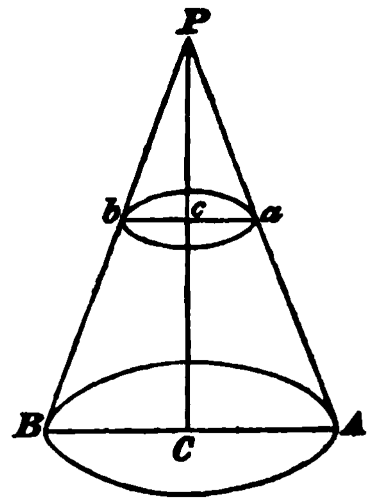
$$r\left(\frac{\sqrt{3} - \sqrt{2}}{\sqrt{n}}\right), \text{ and } r\left(\frac{\sqrt{2} - 1}{\sqrt{n}}\right),$$

beginning with the outermost.

From these results a neat geometrical construction may be derived.

Ex. 10. Find where a section must be made parallel to the base of a cone, that the two parts may be in a given ratio.

Let $AC = r$, $CP = h$, $Pc = x$, and let the upper part be to the lower part as $m : 1$:



therefore, the whole cone : the frustum :: $m + 1 : 1$;

$$\text{now } \frac{ac}{r} = \frac{x}{h}, \quad \therefore \ ac = \frac{rx}{h}: \text{ and } Cc = h - x:$$

therefore, the volume of the frustum

$$= \tfrac{1}{3}\pi(h - x)\left(r^2 + \frac{r^2 x}{h} + \frac{r^2 x^2}{h^2}\right) = \tfrac{1}{3}\frac{\pi r^2}{h^2}(h - x)(h^2 + hx + x^2):$$

and the volume of the cone $= \frac{1}{3}\pi r^2 h$:

$$\text{whence, } (m + 1)(h - x)(h^2 + hx + x^2) = h^3,$$

is the equation for finding x:

$$\text{that is, } (m + 1)(h^3 - x^3) = h^3, \text{ and } \therefore \ x = h\sqrt[3]{\frac{m}{m + 1}}.$$

If $m = 1$, or the volume of the cone be bisected, we have $x = \dfrac{h}{\sqrt[3]{2}}$, as it manifestly ought to be.

On this subject the Student may consult *Bonnycastle's* Algebra, and *Simpson's* Select Exercises for additional Examples.

APPENDIX I.

MISCELLANEOUS THEOREMS AND PROBLEMS, WITH THEIR SOLUTIONS.

1. We will now present to the student a few additional articles connected with the preceding pages, a great portion of which were incorporated into the text of the last edition of this work, but not being essential for a correct apprehension of the principles of Algebra, they are now removed out of his way, so that the progress he has made, or the leisure he may possess, will direct him to their perusal or not.

FUNDAMENTAL OPERATIONS.

2. The chief distinctions between Arithmetical and Symbolical Algebra, consist in the following particulars:

In Arithmetical Algebra, the quantities upon which operations are performed are always supposed to be numbers or their representations by means of general symbols the letters of the alphabet, whereas, in Symbolical Algebra, they may be any kinds of quantities whatever.

In Arithmetical Algebra, the signs made use of, are merely symbols *of operation*, whereas, in Symbolical Algebra, they are frequently supposed to denote *affections* or *qualities* inherent in the symbols to which they are prefixed.

Arithmetical Algebra is grounded entirely upon axioms arising from the nature of number; but Symbolical Algebra is a science which treats of all kinds of quantities as influenced by the affections or qualities of the symbols employed, and its principles are *assumed* to be such that its results are always accordant with those of arithmetic as far as it goes.

On this subject, see Professor Peacock's *Algebra*, where the distinction is fully and clearly pointed out.

3. In article (35), it has been shewn that $\dfrac{x^m - a^m}{x - a}$ is always a complete quotient, whatever positive whole number m may be: and in precisely the same manner, if we divide $x^m - a^m$ by $x + a$, the remainder will always be of the form $\pm a^n x^{m-n} - a^m$, which cannot become $= 0$ when $n = m$, unless the first term be positive, and this is the case only when n is an even number: that is, $\dfrac{x^m - a^m}{x + a}$ is an exact quotient when m is even, but not when m is odd.

Similarly, $\dfrac{x^m + a^m}{x + a}$ may be shewn to give a complete quotient when m is odd, but not when m is even; and that $\dfrac{x^m + a^m}{x - a}$ is never a complete quotient, as expressed in general symbols.

These results are easily established by means of the expansion of the binomial $(x \pm a)^m$.

4. COR. 1. Since $x^m - px^{m-1} + qx^{m-2} - \text{\&c.} + tx - u$, may be written in the form

$$(x^m - a^m) - p(x^{m-1} - a^{m-1}) + q(x^{m-2} - a^{m-2}) - \text{\&c.} + t(x - a)$$
$$+ a^m - pa^{m-1} + qa^{m-2} - \text{\&c.} + ta - u,$$

and every binomial quantity in the first line is divisible by $x - a$, it follows that the remainder not involving x, will be

$$a^m - pa^{m-1} + qa^{m-2} - \text{\&c.} + ta - u :$$

and of this, the third example of article (34) is a particular case.

5. COR. 2. Hence, we shall have also,

$$\frac{(1 + v)^m - 1}{(1 + v) - 1} = 1 + (1 + v) + (1 + v)^2 + (1 + v)^3 + \text{\&c. to } m \text{ terms:}$$

$$\therefore\ (1+v)^m = 1 + v + v\left\{(1+v) + (1+v)^2 + \&\text{c. to } (m-1) \text{ terms}\right\}$$
$$= 1 + mv + Bv^2 + Cv^3 + \&\text{c.}:$$

which proves that the coefficient of the second term of the expansion of $(1+v)^m$ is equal to the index, when it is a positive whole number, as in article (244).

6. We have seen in (4) of article (13), that $a^0 = 1$, and it may not be improper to explain how this happens to be the case, from simple principles.

If we put down 1, and perform successive multiplications and divisions of it by a, we shall have the following scheme:

$$\&\text{c.},\ \frac{1}{a^4},\ \frac{1}{a^3},\ \frac{1}{a^2},\ \frac{1}{a},\ 1,\ a,\ a^2,\ a^3,\ a^4,\ \&\text{c.}:$$

which may also be written in the form,

$$\&\text{c.},\ a^{-4},\ a^{-3},\ a^{-2},\ a^{-1},\ 1,\ a^1,\ a^2,\ a^3,\ a^4,\ \&\text{c.}:$$

and from this we derive the conclusions below:

$$a^1 = 1 \times a:$$
$$a^2 = 1 \times a \times a:$$
$$a^3 = 1 \times a \times a \times a:$$
$$a^4 = 1 \times a \times a \times a \times a:\ \&\text{c}:$$
$$a^{-1} = 1 \div a:$$
$$a^{-2} = (1 \div a) \div a:$$
$$a^{-3} = \left\{(1 \div a) \div a\right\} \div a:$$
$$a^{-4} = \left[\left\{(1 \div a) \div a\right\} \div a\right] \div a:\ \&\text{c.}:$$

so that the index in reality denotes how many successive multiplications or divisions of 1 by the quantity a are implied: and consequently when *no* multiplication or division by a is supposed to take place, we shall have a^0 equivalent to 1, consistently with this view of the subject. Also, the cipher or 0, being the connecting link between positive and negative quantities, it is found convenient to consider it subject to the same operations as any other symbol: and accordingly we shall have $0^0 = 1$, by the same notation, for the scheme will then be,

$$\&\text{c.},\ \frac{1}{0^4},\ \frac{1}{0^3},\ \frac{1}{0^2},\ \frac{1}{0^1},\ 1,\ 0^1,\ 0^2,\ 0^3,\ 0^4,\ \&\text{c.}$$

7. The square of every number consisting of n digits, will contain either $2n - 1$, or $2n$ digits.

For, a number consisting of n digits must either be equal to 10^{n-1}, or lie between 10^{n-1} and 10^n: and consequently its square must either be 10^{2n-2}, or lie between 10^{2n-2} and 10^{2n}: that is, the square must comprise either $2n - 1$, or $2n$ digits, since 10^{2n-2} and 10^{2n} contain exactly $2n - 1$ and $2n + 1$ digits respectively.

Hence also conversely, the square root of a number consisting of either $2n - 1$ or $2n$ digits, will contain n digits.

Similarly, the cube of every number consisting of n digits will contain either $3n - 2$, $3n - 1$, or $3n$ digits.

Whence the rules for pointing in the extraction of the square and cube roots of numbers.

8. To investigate the form of the square of

$$a + b + c + d + \&c. + l.$$

Regarding $b + c + d + \&c. + l$, as if it were represented by a single symbol, we have

$$\{a + (b + c + d + \&c. + l)\}^2$$
$$= a^2 + 2a\,(b + c + d + \&c. + l) + (b + c + d + \&c. + l)^2:$$

$$\text{similarly, } \{b + (c + d + \&c. + l)\}^2$$
$$= b^2 + 2b\,(c + d + \&c. + l) + (c + d + \&c. + l)^2:$$

$$\text{also, } \{c + (d + \&c. + l)\}^2$$
$$= c^2 + 2c\,(d + \&c. + l) + (d + \&c. + l)^2, \ \&c.:$$

whence will be obtained by substitution,

$$\begin{aligned}(a + b + c + d + \&c. + l)^2 &= a^2 + 2a\,(b + c + d + \&c. + l)\\ &+ b^2 + 2b\,(c + d + \&c. + l)\\ &+ c^2 + 2c\,(d + \&c. + l)\\ &+ \&c.\end{aligned}$$

that is, the square of the sum of any number of quantities, is always equal to the sum of their squares, augmented by *twice* the sum of all the products that can be formed by taking two of them together.

9. Cor. Hence we may arrange this result in a different form: for

$$(a + b + c + d + \&c. + l)^2$$
$$= a^2 + (2a + b)\,b + \{2(a + b) + c\}\,c$$
$$+ \{2(a + b + c) + d\}\,d + \&c.:$$

from which the rule for the extraction of the square root given in article (38) is immediately derived, the first term, the first two terms, the first three terms, &c., being the complete squares of a, $a + b$, $a + b + c$, &c.

10. By a similar process, we shall have

$$(a + b + c + \&c. + l)^3$$
$$= a^3 + (3a^2 + 3ab + b^2)\,b + \{3(a + b)^2 + 3(a + b)\,c + c^2\}\,c$$
$$+ \{3(a + b + c)^2 + 3(a + b + c)\,d + d^2\}\,d + \&c.:$$

from which a rule for the extraction of the cube root is immediately obtained.

Ex. Extract the cube root of 122615.327232.

$$\begin{array}{rll}
 & 122615\,.\,327232\ (49\,.\,68 & \\
 & 64 & \\
5961) & 58615 & 3a^2 = 4800 \\
 & 53649 & \\
729156) & 4966327 & 3(a + b)^2 = 720300 \\
 & 4374936 & \\
73923904) & 591391232 & 3(a + b + c)^2 = 73804800 \\
 & 591391232 &
\end{array}$$

where the trial figures in the root are determined by means of the quantities on the right hand, and the complete divisors are found as follows:

$$a = 40, \quad b = 9:$$
$$\therefore\ 3a^2 = 4800, \quad 3a + b = 129:$$
$$\therefore\ 3ab + b^2 = 1161:$$
$$\text{whence, } 3a^2 + 3ab + b^2 = 5961:$$
$$a + b = 49, \quad c = .6:$$
$$\therefore\ 3(a+b)^2 = 7203.00, \quad 3(a+b) + c = 14.76:$$
$$\therefore\ 3(a+b)c + c^2 = 88.56:$$
$$\text{whence, } 3(a+b)^2 + 3(a+b)c + c^2 = 7291.56:$$
$$a + b + c = 49.6, \quad d = .08:$$
$$\therefore\ 3(a+b+c)^2 = 7380.4800, \quad 3(a+b+c) + d = 148.88:$$
$$\therefore\ 3(a+b+c)d + d^2 = 11.9104,$$
$$\text{whence, } 3(a+b+c)^2 + 3(a+b+c)d + d^2 = 7392.3904.$$

11. To extract any root of a compound algebraical quantity.

Since, $(a + x)^m = a^m + ma^{m-1}x +$ &c., it is obvious that when the quantities are properly arranged and the first term of the root is found, the second term of the m^{th} root will be obtained by dividing the second term of the proposed quantity by ma^{m-1}, or by m times the first term raised to the $(m-1)^{\text{th}}$ power: and if the terms in the root thus obtained be raised to the m^{th} power and the result be subtracted from the quantity proposed, and the process be repeated, any root of a compound quantity may be determined.

Ex. 1. Extract the cube root of $a^3 - 3a^2b + 3a^2 + 3ab^2 - 6ab + 3a - b^3 + 3b^2 - 3b + 1$.

$$
\begin{array}{l}
a^3 - 3a^2b + 3a^2 + 3ab^2 - 6ab + 3a - b^3 + 3b^2 - 3b + 1\ (\ a - b + 1 \\
a^3 \\
\hline
3a^2)\ -3a^2b \\
\hline
a^3 - 3a^2b + 3ab^2 - b^3 = (a-b)^3 \\
\hline
3a^2)\ 3a^2 \\
\hline
a^3 - 3a^2b + 3a^2 + 3ab^2 - 6ab + 3a - b^3 + 3b^2 - 3b + 1 = (a - b + 1)^3. \\
\hline
\end{array}
$$

Ex. 2. Find the fifth root of $x^5 - 10x^4y + 40x^3y^2 - 80x^2y^3 + 80xy^4 - 32y^5$.

$$\begin{array}{l} x^5 - 10x^4y + 40x^3y^2 - 80x^2y^3 + 80xy^4 - 32y^5 \,(\, x - 2y \\ x^5 \\ \overline{} \\ 5x^4\,)\, - 10x^4y \\ \hline x^5 - 10x^4y + 40x^3y^2 - 80x^2y^3 + 80xy^4 - 32y^5 = (x-2y)^5. \\ \hline \end{array}$$

12. The purport of the observations made at the end of article (53) may be expressed by means of symbols, in the following form.

Let A and Bb be two compound quantities, of which the highest common factor is required, A not being of lower dimensions than Bb: and let

Aa be divided by B, and leave a remainder Cc:

$B\beta$ .... C, ... Dd:

&c.

where b, c, &c., a, β, γ, &c., contain no factor common to A and B: then, if the operation be continued till the remainder $= 0$, the last divisor, if it contain no factor common to a, β, γ, &c., will be the highest common factor of A and Bb.

For, if the quotients be P, Q, R, and the last divisor D, we shall have

$$\frac{Aa}{B} = P + \frac{Cc}{B}, \text{ or } Aa = PB + Cc:$$

$$\frac{B\beta}{C} = Q + \frac{Dd}{C}, \text{ or } B\beta = QC + Dd:$$

$$\frac{C\gamma}{D} = R, \text{ or } C\gamma = DR:$$

whence, D is a divisor of $C\gamma$, and $\therefore$ of C:

$\therefore$ D is a divisor of QC and $QC + Dd$ or B:

$\therefore$ D is a divisor of PB and $PB + Cc$ or A:

that is, D is a common factor of A and Bb:

also, since $Aa - PB = Cc$:

and $B\beta - QC = Dd$:

every common factor of A and Bb is a common factor of Aa and PB:

and $\therefore$ of $Aa - PB$ or Cc:

and $\therefore$ of C and $B\beta - QC$, and $\therefore$ of D:

that is, since no common factor of A and Bb is greater than D, and D is a common factor of A and Bb, it follows that D is the highest common factor of A and Bb.

It will readily be seen in what parts of this process the considerations of article (51) and the one above referred to, are introduced.

PROPOSITIONS IN FRACTIONS.

13. It sometimes happens that when a particular value is assigned to one of the symbols involved in the terms of fraction, the result appears in the indeterminate form $\frac{0}{0}$, from which no determinate value can be inferred: but this peculiarity being the consequence of some common factor of the numerator and denominator becoming equal to 0, it is manifest that the true value will be obtained by divesting them of such factor determined by the rule of article (56).

Ex. 1. Find the value of the fraction $\dfrac{x^3 - 1}{x^3 - 2x^2 + 2x - 1}$, when $x = 1$.

Here, $$\frac{x^3 - 1}{x^3 - 2x^2 + 2x - 1} = \frac{(x-1)(x^2 + x + 1)}{(x-1)(x^2+x+1) - 2x(x-1)}$$

$$= \frac{x^2 + x + 1}{x^2 - x + 1} = \frac{3}{1} = 3, \text{ when } x = 1.$$

Ex. 2. Find the values of x which will cause the fraction $\dfrac{x^3 - 2ax^2 - a^2x + 2a^3}{x^3 - ax^2 - 4a^2x + 4a^3}$ to assume the form $\dfrac{0}{0}$, and the corresponding values of the fraction.

By the usual method, the common factor of the numerator and denominator is found to be $x^2 - 3ax + 2a^2$,

so that $$\frac{x^3 - 2ax^2 - a^2x + 2a^3}{x^3 - ax^2 - 4a^2x + 4a^3} = \frac{(x+a)(x^2 - 3ax + 2a^2)}{(x+2a)(x^2 - 3ax + 2a^2)}:$$

whence, if $x^2 - 3ax + 2a^2 = 0$, which gives $x = a$ and $x = 2a$, the fraction assumes the form $\dfrac{0}{0}$; but at the same time, in its simplest terms, it is $= \dfrac{x+a}{x+2a}$, whose values, when $x = a$ and $x = 2a$, are $\frac{2}{3}$ and $\frac{3}{4}$.

14. If $\dfrac{ad - bc}{a - b - c + d} = \dfrac{ac - bd}{a - b + c - d}$, it is required to prove that $a + b = c + d$.

Here, $$\frac{ad - bc}{(a-b) - (c-d)} = \frac{ac - bd}{(a-b) + (c-d)}:$$

$$\therefore\ ad(a-b) - bc(a-b) + ad(c-d) - bc(c-d)$$
$$= ac(a-b) - bd(a-b) - ac(c-d) + bd(c-d):$$

whence, $(ad + bd - ac - bc)(a-b) = (bd + bc - ac - ad)(c-d)$,

$$\text{or}\ (a+b)(d-c)(a-b) = (c+d)(b-a)(c-d):$$

$$\therefore\ a + b = c + d.$$

Hence also, $ad - bc = \frac{1}{4}(a + b + c + d)(a - b - c + d)$,

and $ac - bd = \frac{1}{4}(a + b + c + d)(a - b + c - d)$:

$$\therefore\ \frac{ad - bc}{a - b - c + d} = \frac{ac - bd}{a - b + c - d} = \frac{1}{4}(a + b + c + d).$$

15. Shew that the value of the fraction $\frac{a+b+c}{p+q+r}$ lies between those of the greatest and least of the fractions $\frac{a}{p}$, $\frac{b}{q}$ and $\frac{c}{r}$.

Let $\frac{a}{p}$ be the greatest, and $\frac{c}{r}$ the least of the fractions:

$$\text{then, } \frac{a+b+c}{p+q+r} - \frac{c}{r} = \frac{(a+b)r-(p+q)c}{r(p+q+r)}:$$

$$\text{and } \frac{a}{p} - \frac{a+b+c}{p+q+r} = \frac{(q+r)a-(b+c)p}{p(p+q+r)}:$$

$$\text{but } \frac{a}{p} > \frac{c}{r}, \text{ and } \frac{b}{q} > \frac{c}{r}:$$

$$\therefore ar > pc,\ br > qc, \text{ and } (a+b)r > (p+q)c:$$

or $(a+b)r-(p+q)c$ is a positive quantity:

$$\text{also, } \frac{b}{q} < \frac{a}{p}, \text{ and } \frac{c}{r} < \frac{a}{p}:$$

$$\therefore bp < qa,\ cp < ra, \text{ and } (b+c)p < (q+r)a:$$

or $(q+r)a-(b+c)p$ is a positive quantity:

whence it follows immediately that the value of $\frac{a+b+c}{p+q+r}$ is intermediate in magnitude to those of $\frac{a}{p}$ and $\frac{c}{r}$.

Similarly of more fractions.

PROPOSITIONS IN SURDS.

16. If the m^{th} root of a quadratic surd can be extracted, it will be of the form $\sqrt{x}+\sqrt{y}$.

$$\text{For, } (\sqrt{x}+\sqrt{y})^m = x^{\frac{m}{2}} + mx^{\frac{m-1}{2}}\sqrt{y} + \frac{m(m-1)}{1 \cdot 2}x^{\frac{m-2}{2}}y + \&c.:$$

and therefore when m is odd, the odd terms are irrational, involving $\sqrt{x}$, and the even terms are irrational, involving $\sqrt{y}$: whence, denoting these by $\sqrt{a}$ and $\sqrt{b}$ respectively, we have

$(\sqrt{x}+\sqrt{y})^m=\sqrt{a}+\sqrt{b}$, and $\therefore \sqrt[m]{\sqrt{a}+\sqrt{b}}=\sqrt{x}+\sqrt{y}$: also, when m is even, the odd terms will be rational, and the even terms irrational involving $\sqrt{y}$, so that $(\sqrt{x}+\sqrt{y})^m$ is of the form $a+\sqrt{b}$, and $\therefore \sqrt[m]{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}$.

From this it appears that the m^{th} root of $\sqrt{a}+\sqrt{b}$ can be expressed in the form of a binomial quadratic surd $\sqrt{x}+\sqrt{y}$, only when m is an odd number: also, that the m^{th} root of $a+\sqrt{b}$ may be expressed in the form $\sqrt{x}+\sqrt{y}$ when m is an even number, and in the form $x+\sqrt{y}$ whether m be even or odd.

17. When the square root of $a+\sqrt{b}+\sqrt{c}+\sqrt{d}$ can be exhibited in the form $\sqrt{\alpha}+\sqrt{\beta}+\sqrt{\gamma}$: find the relation between a, b, c, d.

Here, $a+\sqrt{b}+\sqrt{c}+\sqrt{d}=(\sqrt{\alpha}+\sqrt{\beta}+\sqrt{\gamma})^2$

$$=\alpha+\beta+\gamma+2\{\sqrt{\alpha\beta}+\sqrt{\alpha\gamma}+\sqrt{\beta\gamma}\}:$$

whence, $\alpha+\beta+\gamma=a$, $2\sqrt{\alpha\beta}=\sqrt{b}$, $2\sqrt{\alpha\gamma}=\sqrt{c}$,

and $2\sqrt{\beta\gamma}=\sqrt{d}$:

from which α, β, γ may be eliminated as follows:

$$4\alpha\sqrt{\beta\gamma}=\sqrt{bc},\quad 4\beta\sqrt{\alpha\gamma}=\sqrt{bd},\ \text{and}\ 4\gamma\sqrt{\alpha\beta}=\sqrt{cd}:$$

$$\therefore\ 64\alpha^2\beta^2\gamma^2=bcd:\ \text{and}\ 2abcd=128(\alpha+\beta+\gamma)\alpha^2\beta^2\gamma^2$$

$$=128\alpha^3\beta^2\gamma^2+128\alpha^2\beta^3\gamma^2+128\alpha^2\beta^2\gamma^3$$

$$=(b^3c^3d)^{\frac{1}{2}}+(b^3d^3c)^{\frac{1}{2}}+(c^3d^3b)^{\frac{1}{2}}=(bc+bd+cd)(bcd)^{\frac{1}{2}},$$

$$\therefore\ 2a(bcd)^{\frac{1}{2}}=bc+bd+cd,$$

which is the relation required.

In $9+2\sqrt{3}+2\sqrt{5}+2\sqrt{15}$, this criterion is satisfied, and the square root $=1+\sqrt{3}+\sqrt{5}$.

18. By effecting the operation of Involution, and indicating the reverse one of Evolution, surds may be transformed into others of different forms.

Thus, since $(x-y+\sqrt{2xy-y^2})^2 = x^2 + 2(x-y)\sqrt{2xy-y^2}$,

we have $x-y+\sqrt{2xy-y^2} = \{x^2 + 2(x-y)\sqrt{2xy-y^2}\}^{\frac{1}{2}}$.

$$\text{Again,} \quad \{(a+b^{\frac{1}{2}})^{\frac{1}{4}} + (a-b^{\frac{1}{2}})^{\frac{1}{4}}\}^2$$

$$= (a+b^{\frac{1}{2}})^{\frac{1}{2}} + (a-b^{\frac{1}{2}})^{\frac{1}{2}} + 2(a^2-b)^{\frac{1}{4}}:$$

$$\text{and} \quad (a+b^{\frac{1}{2}})^{\frac{1}{2}} + (a-b^{\frac{1}{2}})^{\frac{1}{2}} = \{2a + 2(a^2-b)^{\frac{1}{2}}\}^{\frac{1}{2}}:$$

$$\therefore \quad (a+b^{\frac{1}{2}})^{\frac{1}{4}} + (a-b^{\frac{1}{2}})^{\frac{1}{4}} = [\{2a + 2(a^2-b)^{\frac{1}{2}}\}^{\frac{1}{2}} + 2(a^2-b)^{\frac{1}{4}}]^{\frac{1}{2}}.$$

A similar expression holds for $(a+b^{\frac{1}{2}})^{\frac{1}{2^m}} + (a-b^{\frac{1}{2}})^{\frac{1}{2^m}}$.

19. For every compound surd, there exists another compound surd, which being multiplied by it, will give a rational product: thus, the compound surd $\sqrt{a}+\sqrt{b}$ being multiplied by the compound surd $\sqrt{a}-\sqrt{b}$, gives the rational product $a-b$: and the following process contains the general investigation of the multiplier, which will rationalize any binomial surd whatever.

Since, $\dfrac{x^m-y^m}{x-y} = x^{m-1} + x^{m-2}y + \&c. + xy^{m-2} + y^{m-1}$, m being a positive whole number equal to the number of terms:

$$\therefore \quad (x-y)(x^{m-1} + x^{m-2}y + \&c. + xy^{m-2} + y^{m-1}) = x^m - y^m:$$

whence, if x and y represent any two surds, the latter member will manifestly become rational whenever m is assumed of such a magnitude as to render both x^m and y^m rational: and the rationalizing multiplier will be

$$x^{m-1} + x^{m-2}y + \&c. + xy^{m-2} + y^{m-1}.$$

If the sign of the latter surd be positive, we have

$$(x+y)(x^{m-1} - x^{m-2}y + \&c. \mp xy^{m-2} \pm y^{m-1}) = x^m \pm y^m,$$

where the upper or lower sign is to be used according as m is odd or even, and the rationalizing multiplier will be

$$x^{m-1} - x^{m-2}y + \&c. \mp xy^{m-2} \pm y^{m-1}.$$

Ex. 1. Required the factor which will render $a^{\frac{3}{4}} - b^{\frac{3}{4}}$ a rational quantity.

Here, m is manifestly equal to 4, and the multiplier will be

$$a^{\frac{3}{4}\times 3} + a^{\frac{3}{4}\times 2}\, b^{\frac{3}{4}\times 1} + a^{\frac{3}{4}\times 1}\, b^{\frac{3}{4}\times 2} + b^{\frac{3}{4}\times 3},$$

$$\text{or } a^{\frac{9}{4}} + a^{\frac{3}{2}} b^{\frac{3}{4}} + a^{\frac{3}{4}} b^{\frac{3}{2}} + b^{\frac{9}{4}},$$

and the rationalized result is $a^3 - b^3$.

Ex. 2. What is the surd multiplier requisite to rationalize $a^{\frac{1}{2}} + b^{\frac{1}{3}}$?

Here, $m = 6$, and the required multiplier will be

$$a^{\frac{5}{2}} - a^2 b^{\frac{1}{3}} + a^{\frac{3}{2}} b^{\frac{2}{3}} - ab + a^{\frac{1}{2}} b^{\frac{4}{3}} - b^{\frac{5}{3}},$$

and the resulting rational quantity is $a^3 - b^2$.

20. Cor. By continuing the process of the article, the same effects may be produced upon surds consisting of three or more terms.

Thus, let $a^{\frac{1}{2}} + b^{\frac{1}{2}} + c^{\frac{1}{2}}$ be the quantity proposed:

$$\text{then, } \left\{(a^{\frac{1}{2}} + b^{\frac{1}{2}}) + c^{\frac{1}{2}}\right\} \left\{(a^{\frac{1}{2}} + b^{\frac{1}{2}}) - c^{\frac{1}{2}}\right\}$$

$$= (a^{\frac{1}{2}} + b^{\frac{1}{2}})^2 - (c^{\frac{1}{2}})^2 = (a + b - c) + 2(ab)^{\frac{1}{2}}:$$

$$\text{again, } \left\{(a + b - c) + 2(ab)^{\frac{1}{2}}\right\} \left\{(a + b - c) - 2(ab)^{\frac{1}{2}}\right\}$$

$$= (a + b - c)^2 - 4ab = a^2 + b^2 + c^2 - 2(ab + ac + bc),$$

which is a rational quantity, the factors requisite to produce it being

$$a^{\frac{1}{2}} + b^{\frac{1}{2}} - c^{\frac{1}{2}}, \text{ and } a + b - c - 2(ab)^{\frac{1}{2}}.$$

21. The values of what are called *Continued Surds* may always be found as in the following examples.

Ex. 1. Find the value of

$$\sqrt{a^2 + b^2 + \sqrt{a^2 + b^2 + \sqrt{\&\text{c. } in\ infinitum.}}}$$

If x be the value required, we shall manifestly have

$$x^2 = a^2 + b^2 + x, \text{ or } x^2 - x = a^2 + b^2:$$

whence is found $x = \frac{1}{2}\{1 + \sqrt{4(a^2 + b^2) + 1}\}$.

If $a^2 + b^2 = m(m - 1)$, we have

$$x = \tfrac{1}{2}\{1 + \sqrt{4m^2 - 4m + 1}\}$$
$$= \tfrac{1}{2}\{1 + 2m - 1\} = m:$$

also, if $a^2 + b^2 = m^n(m^n - 1)$, then $x = m^n$:

from which it appears that the continued surd corresponding to the latter value of $a^2 + b^2$ is equal to the n^{th} power of that corresponding to the former.

Ex. 2. Required a finite expression for

$$\sqrt{a\sqrt{a\sqrt{\&\text{c. } in\ infinitum.}}}$$

Here, we have $x^2 = ax$, and $\therefore x = a$.

IMAGINARY QUANTITIES.

22. By actual evolution, it is easily proved that

$$\sqrt{x^2 - 1} = x - \frac{1}{2x} - \frac{1}{8x^3} - \frac{1}{16x^5} - \&\text{c. } in\ infinitum:$$

whence, if $x = \pm 0$, we shall have

$$\sqrt{-1} = 0 \mp \frac{1}{0} \mp \frac{1}{0} \mp \frac{1}{0} \mp \&\text{c.}$$
$$= 0 \mp \infty \mp \infty \mp \infty \mp \&\text{c.}$$

to which no definite arithmetical meaning can be attached: and consequently $\sqrt{-1}$ or $(-1)^{\frac{1}{2}}$ cannot be arithmetically assigned, and not even an approximation can be made to its value.

This circumstance shews that, though it may have arisen from the generalizations of symbolical algebra, its origin and meaning must be looked for in other quantities than

numbers, and the student is referred to Professor Peacock's *Algebra* for an ample elucidation of the subject.

23. By reference to article (124), we may find the real values of x and y which satisfy the equation

$$x + y\sqrt{-1} = \frac{a + b\sqrt{-1}}{a - b\sqrt{-1}}.$$

For,

$$x+y\sqrt{-1} = \frac{(a+b\sqrt{-1})(a+b\sqrt{-1})}{(a-b\sqrt{-1})(a+b\sqrt{-1})} = \frac{a^2-b^2}{a^2+b^2} + \frac{2ab}{a^2+b^2}\sqrt{-1}:$$

$$\text{whence, } x = \frac{a^2 - b^2}{a^2 + b^2}, \text{ and } y = \frac{2ab}{a^2 + b^2}.$$

MISCELLANEOUS EQUATIONS.

24. The following equations, possessing certain peculiarities in their characters, will suggest to the student some additional information as to their natures and solutions.

Ex. 1. Find the value of x from the equation,

$$\frac{3ac}{a+b} + \frac{a^2b}{(a+b)^3} + \frac{(2a+b)bx}{a(a+b)^2} = \frac{3cx}{b} + \frac{x}{a}.$$

$$\text{Here, } \frac{3ac}{a+b} + \frac{a^2b}{(a+b)^3} = \frac{3cx}{b} + \frac{x}{a} - \frac{(2a+b)bx}{a(a+b)^2},$$

$$\text{or } \frac{3ac(a+b)^2 + a^2b}{(a+b)^3} = \frac{\{3ac(a+b)^2 + a^2b\}\, x}{ab(a+b)^2}:$$

$$\text{whence, we find } x = \frac{ab}{a+b}.$$

Ex. 2. Given $\dfrac{a^{\frac{1}{2}} - \{a - (a^2 - ax)^{\frac{1}{2}}\}^{\frac{1}{2}}}{a^{\frac{1}{2}} + \{a - (a^2 - ax)^{\frac{1}{2}}\}^{\frac{1}{2}}} = b$, to find x.

$$\text{Here, } a^{\frac{1}{2}} - \{a - (a^2 - ax)^{\frac{1}{2}}\}^{\frac{1}{2}} = a^{\frac{1}{2}}b + b\{a - (a^2 - ax)^{\frac{1}{2}}\}^{\frac{1}{2}}:$$

$$\therefore\ (1-b)a^{\frac{1}{2}} = (1+b)\{a - (a^2 - ax)^{\frac{1}{2}}\}^{\frac{1}{2}}:$$

$$\text{whence, } a - (a^2 - ax)^{\frac{1}{2}} = \left(\frac{1-b}{1+b}\right)^2 a:$$

$$(a^2 - ax)^{\frac{1}{2}} = \frac{4ab}{(1+b)^2}, \text{ and } \therefore\ x = a\left\{1 - \frac{16b^2}{(1+b)^4}\right\}.$$

Ex. 3. Given $\dfrac{a + x^{\frac{1}{2}}}{a^{\frac{1}{2}} + (a + x^{\frac{1}{2}})^{\frac{1}{2}}} + \dfrac{a - x^{\frac{1}{2}}}{a^{\frac{1}{2}} + (a - x^{\frac{1}{2}})^{\frac{1}{2}}} = a^{\frac{1}{2}}$, to find x.

Here, $\dfrac{a + x^{\frac{1}{2}}}{a^{\frac{1}{2}} + (a + x^{\frac{1}{2}})^{\frac{1}{2}}} = \dfrac{x^{\frac{1}{2}} + a^{\frac{1}{2}}(a - x^{\frac{1}{2}})^{\frac{1}{2}}}{a^{\frac{1}{2}} + (a - x^{\frac{1}{2}})^{\frac{1}{2}}}:$

$$\therefore\ a^{\frac{3}{2}} + x^{\frac{1}{2}}(a - x^{\frac{1}{2}})^{\frac{1}{2}} = x^{\frac{1}{2}}(a + x^{\frac{1}{2}})^{\frac{1}{2}} + a^{\frac{1}{2}}(a^2 - x)^{\frac{1}{2}}:$$

$$\text{whence, } a^{\frac{1}{2}}\left\{a - (a^2 - x)^{\frac{1}{2}}\right\} = x^{\frac{1}{2}}\left\{(a + x^{\frac{1}{2}})^{\frac{1}{2}} - (a - x^{\frac{1}{2}})^{\frac{1}{2}}\right\}:$$

and dividing both sides by $(a + x^{\frac{1}{2}})^{\frac{1}{2}} - (a - x^{\frac{1}{2}})^{\frac{1}{2}}$, we have

$$a^{\frac{1}{2}}\left\{(a + x^{\frac{1}{2}})^{\frac{1}{2}} - (a - x^{\frac{1}{2}})^{\frac{1}{2}}\right\} = 2x^{\frac{1}{2}}:$$

$$\therefore\ a^2 - a(a^2 - x)^{\frac{1}{2}} = 2x, \text{ and } a^2 - 2x = a(a^2 - x)^{\frac{1}{2}},$$

from which we find immediately $x = \frac{3}{4}a^2$.

In this instance, the value 0 of x which also satisfies the equation, has been passed over by reason of the equal divisions performed in the solution.

Ex. 4. Given $(x^{\frac{1}{2}} + a)^{\frac{1}{5}} - (x^{\frac{1}{2}} - a)^{\frac{1}{5}} = (2a)^{\frac{1}{5}}$, to find x.

$$\text{Here, } x^{\frac{1}{2}} + a - 5(x^{\frac{1}{2}} + a)^{\frac{4}{5}}(x^{\frac{1}{2}} - a)^{\frac{1}{5}} + 10(x^{\frac{1}{2}} + a)^{\frac{3}{5}}(x^{\frac{1}{2}} - a)^{\frac{2}{5}}$$
$$- 10(x^{\frac{1}{2}} + a)^{\frac{2}{5}}(x^{\frac{1}{2}} - a)^{\frac{3}{5}} + 5(x^{\frac{1}{2}} + a)^{\frac{1}{5}}(x^{\frac{1}{2}} - a)^{\frac{4}{5}} - x^{\frac{1}{2}} + a = 2a:$$

$$\therefore\ 5(x - a^2)^{\frac{1}{5}}\left\{(x^{\frac{1}{2}} + a)^{\frac{3}{5}} - (x^{\frac{1}{2}} - a)^{\frac{3}{5}}\right\}$$
$$= 10(x - a^2)^{\frac{2}{5}}\left\{(x^{\frac{1}{2}} + a)^{\frac{1}{5}} - (x^{\frac{1}{2}} - a)^{\frac{1}{5}}\right\}:$$

$$\therefore\ (x^{\frac{1}{2}} + a)^{\frac{3}{5}} - (x^{\frac{1}{2}} - a)^{\frac{3}{5}} = 2(x - a^2)^{\frac{1}{5}}\left\{(x^{\frac{1}{2}} + a)^{\frac{1}{5}} - (x^{\frac{1}{2}} - a)^{\frac{1}{5}}\right\}:$$

but since $(x^{\frac{1}{2}} + a)^{\frac{1}{5}} - (x^{\frac{1}{2}} - a)^{\frac{1}{5}} = (2a)^{\frac{1}{5}}$, we have

$$(x^{\frac{1}{2}} + a)^{\frac{3}{5}} - (x^{\frac{1}{2}} - a)^{\frac{3}{5}} - 3(x - a^2)^{\frac{1}{5}}\left\{(x^{\frac{1}{2}} + a)^{\frac{1}{5}} - (x^{\frac{1}{2}} - a)^{\frac{1}{5}}\right\} = (2a)^{\frac{3}{5}}:$$

$$\therefore\ (x^{\frac{1}{2}} + a)^{\frac{3}{5}} - (x^{\frac{1}{2}} - a)^{\frac{3}{5}} = (2a)^{\frac{3}{5}} + 3(2a)^{\frac{1}{5}}(x - a^2)^{\frac{1}{5}}:$$

whence, $(2a)^{\frac{2}{5}} + 3(2a)^{\frac{1}{5}}(x - a^2)^{\frac{1}{5}} = 2(2a)^{\frac{1}{5}}(x - a^2)^{\frac{1}{5}}$:

that is, $(x - a^2)^{\frac{1}{5}} = -(2a)^{\frac{2}{5}}$, and $\therefore\ x = -3a^2$.

If $x = a^2$, and $\therefore\ x^{\frac{1}{2}} = \pm a$, the equation will be satisfied: but we have not met with this value in the solution above given, in consequence of the equal divisions by $5(x - a^2)^{\frac{1}{5}}$, which being put $= 0$, gives $x = a^2$.

Ex. 5. Given $\sqrt{10x + 11} = 44 - 5x$, to find x.

Proceeding by the ordinary method, we have

$$x^2 - 18x = -77, \text{ and } \therefore\ x = 11 \text{ and } x = 7.$$

Here, $x = 7$ alone satisfies the proposed equation: and in fact, by the process of rationalizing, a new condition has been introduced which did not originally belong to it, but is satisfied by $x = 11$: thus, giving the double sign to $\sqrt{10x + 11}$, the reduced equation comprises both the following

$$\pm\sqrt{10x + 11} = 44 - 5x,$$

in one of which $x = 7$, and in the other $x = 11$.

Ex. 6. Given $2x + \sqrt{x^2 - 7} = 5$, to find x.

Here, $\sqrt{x^2 - 7} = 5 - 2x$, and $\therefore\ x^2 - \dfrac{20}{3}x = -\dfrac{32}{3}$:

which solved gives $x = 4$ and $x = 2\frac{2}{3}$.

Upon trial it will appear that neither of the values of x here found satisfies the equation, and as there is no other mode of proceeding to which recourse may be had, it is evident that a congruent answer to the equation proposed cannot be ascertained. No theory can prove, nor indeed is it true, that every equation has even one root, unless it be expressed in a rational form, and the instance here selected evinces it, the roots above found belonging to the equation $2x - \sqrt{x^2 - 7} = 5$, which when rationalized assumes the same form as the one proposed.

See an Article by *Mr Horner* in the London and Edinburgh Philosophical Magazine for January, 1836.

Ex. 7. Given $x + \sqrt{a^2 + bx} = a$, to find x.

This equation gives $x^2 - (2a + b)\,x = 0$:

$$\therefore\ x^2 - (2a+b)x + \frac{(2a+b)^2}{4} = \frac{(2a+b)^2}{4}:$$

$$\therefore\ x - \frac{2a+b}{2} = \pm\,\frac{2a+b}{2},\ \text{ and } x = 2a + b \text{ or } 0.$$

The reduced equation is $\{x - (2a + b)\}\,x = 0$, which gives $x = 0$, and $x - (2a + b) = 0$, or $x = 2a + b$.

Ex. 8. Given $\dfrac{x + \sqrt{x^2 - a^2}}{x - \sqrt{x^2 - a^2}} = \dfrac{x}{a}$, to find x.

$$\text{Here, } ax + a\sqrt{x^2 - a^2} = x^2 - x\sqrt{a^2 - x^2}:$$

$$\therefore\ (a + x)\sqrt{x^2 - a^2} = x^2 - ax = x\,(x - a):$$

whence, dividing both sides by $\sqrt{x - a}$, we have

$$(a + x)^{\frac{3}{2}} = x\,(x - a)^{\frac{1}{2}},\ \text{ and } \therefore\ x^2 + \tfrac{3}{4}\,ax = -\,\tfrac{1}{4}\,a^2,$$

which solved gives $x = \frac{1}{8}\,a\,(-3 \pm \sqrt{-7})$.

By this process no real value of x has been obtained, but if we resume the equation at the step where

$$(a + x)\sqrt{x^2 - a^2} = a\,(x - a),$$

and put it in the form

$$\{(x + a)^{\frac{3}{2}} - a\,(x - a)^{\frac{1}{2}}\}\,(x - a)^{\frac{1}{2}} = 0,$$

we shall have likewise $(x - a)^{\frac{1}{2}} = 0$, which gives $x = a$, a real quantity.

Ex. 9. Given $x^2 + \dfrac{1}{x^2} = \dfrac{35x^2 - 62x + 35}{6x}$, to find x.

$$\text{Here, } x^2 + \frac{1}{x^2} = \frac{35}{6}\left(x + \frac{1}{x}\right) - \frac{31}{3}:$$

$$\therefore \left(x+\frac{1}{x}\right)^2 - \frac{35}{6}\left(x+\frac{1}{x}\right) = -\frac{25}{3}:$$

whence, completing the square, &c., we find

$$x+\frac{1}{x} = \frac{10}{3}, \text{ and } x+\frac{1}{x} = \frac{5}{2},$$

the former of which gives $x = 3$ and $x = \frac{1}{3}$, and from the latter we have $x = 2$ and $x = \frac{1}{2}$.

Equations whose roots are of the form a, $\frac{1}{a}$, b, $\frac{1}{b}$, &c., are termed *Reciprocal* Equations from the forms of the roots: also, because, after the proper reductions, the coefficients of the terms from the beginning and end are equal, they are sometimes known by the name of *Recurring* Equations: and it will appear that they may always be reduced to equations of half the number of dimensions when they are of an even order; also, that the odd root is 1 or -1, according as the last term is negative or positive, when the number of dimensions is odd.

Ex. 10. Given $x^4 - 5x^3 + 6x^2 - 5x + 1 = 0$, to find x.

Here, dividing every term by x^2, we have

$$x^2 - 5x + 6 - \frac{5}{x} + \frac{1}{x^2} = 0, \text{ or } \left(x^2+\frac{1}{x^2}\right) - 5\left(x+\frac{1}{x}\right) + 6 = 0:$$

whence, assuming $y = x + \frac{1}{x}$, and $\therefore x^2 + \frac{1}{x^2} = y^2 - 2$, the equation becomes

$$y^2 - 5y + 4 = 0, \text{ from which } y = 4 \text{ and } y = 1:$$

$$\text{that is, } x + \frac{1}{x} = 4, \text{ gives } x = 2 \pm \sqrt{3},$$

$$\text{and } x + \frac{1}{x} = 1, \text{ gives } x = \tfrac{1}{2}(1 \pm \sqrt{-3}).$$

The four roots here found, do not appear in the form a, $\frac{1}{a}$, b, $\frac{1}{b}$, but they are easily made to do so by reduction:

thus, if $a = 2 + \sqrt{3}$, we have

$$\frac{1}{a} = \frac{1}{2+\sqrt{3}} = \frac{2-\sqrt{3}}{(2+\sqrt{3})(2-\sqrt{3})} = \frac{2-\sqrt{3}}{1} = 2 - \sqrt{3}:$$

similarly, if $b = \frac{1}{2}(1 + \sqrt{-3})$, then, $\frac{1}{b} = \frac{1}{2}(1 - \sqrt{-3})$.

Ex. 11. Given $x^5 + 1 = 0$, to find the values of x.

Since $x^5 + 1 = (x+1)(x^4 - x^3 + x^2 - x + 1) = 0$, the equation is satisfied when $x + 1 = 0$, or $x = -1$:

also, $x^4 - x^3 + x^2 - x + 1 = 0$, gives

$$\left(x^2 + \frac{1}{x^2}\right) - \left(x + \frac{1}{x}\right) + 1 = 0,$$

as in the last example: and the values of x will be obtained as before.

Ex. 12. Given $(a+x)^{\frac{1}{4}} + (a-x)^{\frac{1}{4}} = b^{\frac{1}{4}}$, to find x.

$$\text{Here, } (a+x)^{\frac{1}{2}} + 2(a^2 - x^2)^{\frac{1}{4}} + (a-x)^{\frac{1}{2}} = b^{\frac{1}{2}}:$$

$$\therefore \; (a+x)^{\frac{1}{2}} + (a-x)^{\frac{1}{2}} = b^{\frac{1}{2}} - 2(a^2 - x^2)^{\frac{1}{4}}:$$

$$\text{whence, } 2a + 2(a^2 - x^2)^{\frac{1}{2}} = b - 4b^{\frac{1}{2}}(a^2 - x^2)^{\frac{1}{4}} + 4(a^2 - x^2)^{\frac{1}{2}}:$$

$$\therefore \; (a^2 - x^2)^{\frac{1}{2}} - 2b^{\frac{1}{2}}(a^2 - x^2)^{\frac{1}{4}} = a - \tfrac{1}{2}b, \text{ a quadratic form:}$$

$$\text{which gives } x = \pm\left[a^2 - \left\{b^{\frac{1}{2}} \pm (a + \tfrac{1}{2}b)^{\frac{1}{2}}\right\}^4\right]^{\frac{1}{2}}.$$

Ex. 13. Solve the equation $x^3 + 4x^2 - 25x - 28 = 0$.

$$\text{Here, we have } x^2 + 4x - 25 = y, \text{ if } y = \frac{28}{x}:$$

$$\text{which gives } x = -2 \pm \sqrt{y + 29}:$$

whence, if x be a whole number, we must have $y + 29$ a complete square, and this will be the case when $y = -28$, and therefore $x = -1$: also, when $y = -4$, and therefore $x = -7$: and when $y = 7$, and therefore $x = 4$:

that is, the roots are -1, -7 and 4, as will be found upon trial: but this method being purely tentative, cannot be looked

upon as a theoretical solution, although it restricts the value of the unknown quantity within certain limits, which will generally facilitate its discovery.

Ex. 14. If the roots of an equation be a, b, c, such that $x = a$, $x = b$, $x = c$: we have $x - a = 0$, $x - b = 0$, $x - c = 0$: whence, $(x - a)(x - b)(x - c) = 0$, is satisfied by each of the quantities a, b, c: that is, a, b, c are the roots of the equation

$$x^3 - (a + b + c)x^2 + (ab + ac + bc)x - abc = 0.$$

Hence, conversely as in article (143), the coefficient of the second term, with its proper sign, is the sum of the roots with their signs changed: the coefficient of the third term is the sum of the products of every two roots with their signs changed, and the coefficient of the last term is the product of all the roots with their signs changed.

Similar conclusions may be drawn, whatever be the number of dimensions of the equation.

Ex. 15. The roots of the equation $x^3 - 6x^2 + 11x - 6 = 0$, are in arithmetical progression: find them.

Let $\alpha + \beta$, α and $\alpha - \beta$ be the roots:

then $3\alpha = 6$, or $\alpha = 2$, the middle root:

also, $\alpha^2 + \alpha\beta + \alpha^2 - \beta^2 + \alpha^2 - \alpha\beta = 11$, or $3\alpha^2 - \beta^2 = 11$:

whence, $\beta^2 = 3\alpha^2 - 11 = 12 - 11 = 1$, and $\beta = \pm 1$:

$\therefore\ \alpha + \beta = 2 \pm 1 = 3$ or 1, and $\alpha - \beta = 2 \mp 1 = 1$ or 3:

and the three roots are 1, 2 and 3.

Ex. 16. Required the roots of

$$x^4 - 10x^3 + 35x^2 - 50x + 24 = 0,$$

which are in arithmetical progression.

Let $\alpha - 3\beta$, $\alpha - \beta$, $\alpha + \beta$ and $\alpha + 3\beta$, which are in arithmetical progression, represent the roots:

then $4\alpha = 10$, and $\alpha = 2\frac{1}{2}$:

also, $(\alpha^2 - 9\beta^2)(\alpha^2 - \beta^2) = 24$, or $\alpha^4 - 10\alpha^2\beta^2 + 9\beta^4 = 24$,

from which $\beta = \pm\frac{1}{2}$: and $\therefore$ the roots are 1, 2, 3, 4.

Ex. 17. The roots of $x^3 - 7x^2 + 14x - 8 = 0$, are in geometrical progression: find them.

Let $\frac{a}{\beta}$, a and $a\beta$, which are in geometrical progression, denote the roots: then, by example (14), we have

$$\frac{a}{\beta} \times a \times a\beta = 8, \text{ or } a^3 = 8, \text{ and therefore } a = 2:$$

$$\text{also, } \frac{a}{\beta} + a + a\beta = 7, \text{ or } \frac{2}{\beta} + 2 + 2\beta = 7,$$

from which β is found to be ± 2: whence, the roots are 1, 2 and 4.

Ex. 18. Solve the equation $x^4 - 15x^3 + 70x^2 - 120x + 64 = 0$, whose roots are in geometrical progression.

Let $\frac{a}{\beta^3}$, $\frac{a}{\beta}$, $a\beta$ and $a\beta^3$ represent the roots:

$$\text{then, } a^4 = 64, \text{ and therefore } a = \pm 2\sqrt{2}:$$

$$\text{also, } 70 = a^2\left(\beta^4 + 2 + \frac{1}{\beta^4} + \beta^2 + \frac{1}{\beta^2}\right):$$

$$\text{or, } \left(\beta^2 + \frac{1}{\beta^2}\right)^2 + \left(\beta^2 + \frac{1}{\beta^2}\right) = \frac{35}{4},$$

$$\text{which solved gives } \beta = \pm\sqrt{2}, \text{ or } \beta = \pm\frac{1}{\sqrt{2}}:$$

so that the four roots are 1, 2, 4 and 8.

Ex. 19. The roots of $x^3 - 11x^2 + 36x - 36 = 0$, are in harmonical progression: find them.

Let a, β, γ be the roots of the equation:

$$\text{then, } a : \gamma = a - \beta : \beta - \gamma, \text{ or } a\beta - a\gamma = a\gamma - \beta\gamma:$$

$$\therefore a\beta + \beta\gamma = 2a\gamma, \text{ and } a\beta + a\gamma + \beta\gamma = 3a\gamma:$$

$$\text{that is, } 36 = \frac{3a\beta\gamma}{\beta} = \frac{3 \times 36}{\beta}, \text{ or } \beta = 3:$$

$$\text{also, } \alpha + 3 + \gamma = 11, \text{ or } \alpha + \gamma = 8, \text{ and } \alpha\gamma = 12,$$

$$\text{from which we have } \alpha = 2 \text{ and } \gamma = 6:$$

and the three roots are 2, 3 and 6.

If we put $x = \dfrac{1}{y}$ in the proposed equation, the values of y being the reciprocals of those of x will be in arithmetical progression by article (213), and may therefore be found as in example (15): and similarly of others whose roots are thus related.

Ex. 20. To transform the equation $x^3 - px^2 + qx - r = 0$, into another which shall want the second term.

Assume $x = y + h$, and substituting we have

$$\begin{array}{rcllll}
x^3 &=& y^3 + 3hy^2 & + 3h^2y & + h^3 \\
-px^2 &=& \quad - py^2 & - 2phy & - ph^2 \\
+qx &=& & + qy & + qh \\
-r &=& & & - r \\
\hline
\end{array}$$

$$\therefore\; 0 = y^3 + (3h - p)y^2 + (3h^2 - 2ph + q)y + h^3 - ph^2 + qh - r:$$

whence, if h be so assumed that $3h - p = 0$, or $h = \frac{1}{3}p$, the resulting equation becomes

$$y^3 + (3h^2 - 2ph + q)y + h^3 - ph^2 + qh - r = 0,$$

in which the second power of y is not found.

If $h^3 - ph^2 + qh - r = 0$, the equation is reduced to

$$y^3 + (3h^2 - 2ph + q)y = 0, \text{ or } y\left\{y^2 + 3h^2 - 2ph + q\right\} = 0,$$

whose roots are immediately obtained.

If $3h^2 - 2ph + q = 0$, the equation becomes

$$y^3 + h^3 - ph^2 + qh - r = 0,$$

from which y, and thence $x = y + h$ are found.

Since $h = \frac{1}{3}p$, the equation may be reduced immediately whenever $2p^3 - 9pq + 27r = 0$, or $p^2 - 3q = 0$, which are the relations of the coefficients expressed as equations of condition.

Ex. 21. To solve the equation $x^3 - qx + r = 0$.

Let $x = \alpha + \beta$: then by substitution, we have

$$x^3 = (\alpha + \beta)^3 = \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3$$
$$= \alpha^3 + \beta^3 + 3\alpha\beta(\alpha + \beta) = \alpha^3 + \beta^3 + 3\alpha\beta x:$$

$\therefore\ x^3 - 3\alpha\beta x - (\alpha^3 + \beta^3) = 0$, which, being identified with $x^3 - qx + r = 0$, gives

$$3\alpha\beta = q \text{ and } \therefore\ \beta = \frac{q}{3\alpha}: \text{ also, } \alpha^3 + \beta^3 = -r:$$

$$\text{and } \therefore\ \alpha^3 + \frac{q^3}{27\alpha^3} = -r, \text{ or } \alpha^6 + r\alpha^3 = -\frac{q^3}{27}:$$

and this, by the ordinary method, gives

$$\alpha = \left(-\frac{r}{2} \pm \sqrt{\frac{r^2}{4} - \frac{q^3}{27}}\right)^{\frac{1}{3}}:$$

$$\therefore\ \beta = (-r - \alpha^3)^{\frac{1}{3}} = \left(-\frac{r}{2} \mp \sqrt{\frac{r^2}{4} - \frac{q^3}{27}}\right)^{\frac{1}{3}}:$$

$$\text{whence, } x = \left(-\frac{r}{2} \pm \sqrt{\frac{r^2}{4} - \frac{q^3}{27}}\right)^{\frac{1}{3}} + \left(-\frac{r}{2} \mp \sqrt{\frac{r^2}{4} - \frac{q^3}{27}}\right)^{\frac{1}{3}},$$

is one of the roots required.

This is *Cardan's* Solution of a Cubic Equation, and it may be observed here that every cubic equation is reducible to the form of this example by means of example (20).

If $\frac{r^2}{4}$ be greater than $\frac{q^3}{27}$, the value of $\sqrt{\frac{r^2}{4} - \frac{q^3}{27}}$ is a real quantity, and therefore the value of x is also real.

If $\frac{r^2}{4}$ be less than $\frac{q^3}{27}$, the values of α and β are both imaginary, and the value of x is real, as appears from article (263): but since it can only be expressed in the form of an infinite series, this is termed the *irreducible* case of Cardan's Solution, and recourse is usually had to *Trigonometry* in order to avoid it.

Ex. 22. Solve the equation $x^3 - 9x + 28 = 0$.

Let $x = \alpha + \beta$: $\therefore\ x^3 - 3\alpha\beta x - (\alpha^3 + \beta^3) = 0$:

whence, we have $3\alpha\beta = 9$, and $\alpha^3 + \beta^3 = -28$,

from which are obtained $\alpha = -1$, and $\beta = -3$:

and one root $= \alpha + \beta = -4$:

$$\therefore\ x^3 - 9x + 28 = (x + 4)(x^2 - 4x + 7) = 0,$$

which is satisfied also by $x^2 - 4x + 7 = 0$:

and this gives $x = 2 \pm \sqrt{-3}$: so that the three roots of the equation are -4, $2 + \sqrt{-3}$ and $2 - \sqrt{-3}$, one of which is real, and two are imaginary.

Ex. 23. Solve the equation $x^3 - 3x^2 + 4 = 0$.

Agreeably to example (20) put $x = y + 1$, and the equation becomes $y^3 - 3y + 2 = 0$: whence, if $y = \alpha + \beta$, we have

$$y^3 - 3\alpha\beta y - (\alpha^3 + \beta^3) = 0 = y^3 - 3y + 2:$$

$$\therefore\ 3\alpha\beta = 3, \text{ and } \alpha^3 + \beta^3 = -2:$$

from which, $\alpha^3 + 2 + \dfrac{1}{\alpha^3} = 0$, or $\alpha^6 + 2\alpha^3 + 1 = 0$:

and $\therefore\ \alpha^3 + 1 = 0$, or $\alpha = -1$: $\therefore\ \beta = -1$, and $y = -2$:

whence, we have $x = y + 1 = -1$:

$$\therefore\ x^3 - 3x^2 + 4 = (x + 1)(x^2 - 4x + 4) = 0,$$

and the two remaining roots are comprised in

$$x^2 - 4x + 4 = 0, \text{ or } (x - 2)(x - 2) = 0,$$

and are therefore 2 and 2, or are equal.

Ex. 24. To find the three cube roots of 1 and -1.

Let $x = \sqrt[3]{1}$: $\therefore\ x^3 = 1$, and $x^3 - 1 = 0$:

but $x^3 - 1 = (x - 1)(x^2 + x + 1) = 0$,

is satisfied by making $x - 1 = 0$, or $x = 1$: also, by making $x^2 + x + 1 = 0$, which gives $x = \frac{1}{2}\{-1 \pm \sqrt{-3}\}$: that is, the three cube roots of 1 are 1, which is real, and $\frac{1}{2}\{-1 \pm \sqrt{-3}\}$, which are imaginary.

In the same way, the three cube roots of -1 are -1, and $\frac{1}{2}\{1 \pm \sqrt{-3}\}$.

Hence, in the preceding examples, $\alpha + \beta$ admits of *nine* different values: for the values of the cube roots of α^3 are α, $\frac{1}{2}\{-1 + \sqrt{-3}\}\alpha$, and $\frac{1}{2}\{-1 - \sqrt{-3}\}\alpha$, and those of β^3 are β, $\frac{1}{2}\{-1 + \sqrt{-3}\}\beta$, and $\frac{1}{2}\{-1 - \sqrt{-3}\}\beta$: but since the product of $\alpha\beta$ is real, those values must be alone retained, which satisfy this condition: and the admissible values of $\alpha + \beta$ are therefore

$$\alpha + \beta,\ \frac{1}{2}\{-1 + \sqrt{-3}\}\alpha + \frac{1}{2}\{-1 - \sqrt{-3}\}\beta,$$

$$\text{and } \frac{1}{2}\{-1 - \sqrt{-3}\}\alpha + \frac{1}{2}\{-1 + \sqrt{-3}\}\beta,$$

the radical quantity in the values of α and β being affected with different signs, in order that the imaginary quantity may disappear in their product.

In example (22), $\alpha = -1$ and $\beta = -3$: therefore the values of x are $-1 - 3 = -4$:

$$-\frac{1}{2}\{-1 + \sqrt{-3}\} - \frac{3}{2}\{-1 - \sqrt{-3}\} = 2 + \sqrt{-3}:$$

$$\text{and } -\frac{1}{2}\{-1 - \sqrt{-3}\} - \frac{3}{2}\{-1 + \sqrt{-3}\} = 2 - \sqrt{-3}:$$

as there found.

Cardan's Solution is always irreducible, unless two roots are imaginary or equal.

For, let $a + \sqrt{3b}$, $a - \sqrt{3b}$ and c be the roots: then, by example (14), we shall have

$$x^3 - qx + r = \{x - (a + \sqrt{3b})\}\{x - (a - \sqrt{3b})\}(x - c)$$

$$= \{(x - a) - \sqrt{3b}\}\{(x - a) + \sqrt{3b}\}(x - c)$$

$$= \{(x - a)^2 - 3b\}(x - c) = (x^2 - 2ax + a^2 - 3b)(x - c)$$

$$= x^3 - (2a + c)x^2 + (a^2 - 3b + 2ac)x - (a^2 - 3b)c:$$

whence, equating coefficients, we have $2a+c=0$, or $c=-2a$:

$$q=-(a^2-3b+2ac)=-(a^2-3b-4a^2)=3a^2+3b,$$

$$\text{and } r=-(a^2-3b)c=2a^3-6ab:$$

$$\therefore\ \frac{r^2}{4}-\frac{q^3}{27}=(a^3-3ab)^2-(a^2+b)^3$$

$$=-9a^4b+6a^2b^2-b^3=-b(3a^2-b)^2:$$

$$\text{and } \sqrt{\frac{r^2}{4}-\frac{q^3}{27}}=(3a^2-b)\sqrt{-b},$$

which is imaginary, unless b be negative or $=0$, that is, unless two roots be imaginary or equal: and this has appeared to be the case in the preceding applications of the rule.

Hence, if two roots be equal, the relation subsisting among the coefficients will be expressed by $27r^2=4q^3$, as has just been shewn.

Ex. 25. To find when the equation $x^4+px^3+qx^2+rx+s=0$, can be solved as a quadratic.

For x put $y+h$, and the equation becomes

$$\begin{array}{rcccccc}
x^4 = & y^4 & +\,4hy^3 & +\ 6h^2y^2 & +\ 4h^3y & +\,h^4 \\
+\,px^3 = & & py^3 & +\,3phy^2 & +\,3ph^2y & +\,ph^3 \\
+\,qx^2 = & & & qy^2 & +\,2qhy & +\,qh^2 \\
+\,rx = & & & & +\ \ ry & +\,rh \\
+\ \ s = & & & & & +\,s:
\end{array}$$

whence, if $4h+p=0$, and $4h^3+3ph^2+2qh+r=0$, the equation will be reduced to the form $y^4+Py^2+Q=0$, to which the solution of quadratics is immediately applicable:

but $h=-\dfrac{p}{4}$, and therefore, by substitution, we obtain

$$p^3-4pq+8r=0,$$

which expresses the relation subsisting among the coefficients of x, whenever this transformation renders the equation capable of solution by completing the square.

It appears also that by assuming $h = -\frac{p}{4}$, the transformed equation wants the second term, and thus every biquadratic equation is reducible to the form

$$x^4 + qx^2 + rx + s = 0.$$

Ex. 26. To reduce the solution of $x^4 + qx^2 + rx + s = 0$, to the solution of a cubic.

Let $x^4 + qx^2 + rx + s = (x^2 + ex + f)(x^2 - ex + g) = 0$: then, if the values of e, f and g can be determined, the roots of the equation proposed will be found by solving the two quadratics $x^2 + ex + f = 0$, and $x^2 - ex + g = 0$:

$$\text{now, } x^4 + qx^2 + rx + s = \{x^2 + ex + f\}\{x^2 - ex + g\}$$
$$= x^4 + (f + g - e^2)x^2 - (f - g)ex + fg:$$

whence, by equating coefficients, we obtain

$$f + g = q + e^2, \quad f - g = -\frac{r}{e} \text{ and } fg = s:$$

$$\therefore\ 2f = q + e^2 - \frac{r}{e}, \text{ and } 2g = q + e^2 + \frac{r}{e}:$$

$$\therefore\ 4s = 4fg = \left\{(q + e^2) - \frac{r}{e}\right\}\left\{(q + e^2) + \frac{r}{e}\right\}$$
$$= (q + e^2)^2 - \frac{r^2}{e^2} = q^2 + 2qe^2 + e^4 - \frac{r^2}{e^2}:$$
$$\therefore\ e^6 + 2qe^4 + (q^2 - 4s)e^2 - r^2 = 0:$$

and by putting $y = e^2$, we have

$$y^3 + 2qy^2 + (q^2 - 4s)y - r^2 = 0,$$

which is a cubic equation, by means whereof the value of y or e^2 may be found by *Cardan's* Rule: and thus the values of f and g being determined, the roots of the biquadratic will be obtained by means of the two quadratics

$$x^2 + ex + f = 0, \text{ and } x^2 - ex + g = 0.$$

This solution, which is due to *Des Cartes*, will be complete only, when the irreducible case of *Cardan's* Rule is evaded: and that is, when two roots of the equation are real, and two imaginary or equal, as appears hereafter.

If a be a root of the *reducing* cubic equation, the four roots of the equation $x^4 + qx^2 + rx + s = 0$, will be

$$-\tfrac{1}{2}\sqrt{a} \pm \sqrt{-\frac{q}{2} - \frac{a}{4} + \frac{r}{2\sqrt{a}}},$$

$$\text{and } \tfrac{1}{2}\sqrt{a} \pm \sqrt{-\frac{q}{2} - \frac{a}{4} - \frac{r}{2\sqrt{a}}},$$

as appears immediately by putting $e = \sqrt{a}$,

$$f = \tfrac{1}{2}\left\{q + a - \frac{r}{\sqrt{a}}\right\}, \text{ and } g = \tfrac{1}{2}\left\{q + a + \frac{r}{\sqrt{a}}\right\},$$

and then solving the two reducing quadratics.

We may also express the roots of the biquadratic in terms of all the three roots of the reducing cubic.

For, let α^2, β^2, γ^2, be the three roots of the cubic

$$y^3 + 2qy^2 + (q^2 - 4s)\,y - r^2 = 0:$$

$\therefore\ q = -\tfrac{1}{2}(\alpha^2 + \beta^2 + \gamma^2)$, and $r = \alpha\beta\gamma$, by example (14):

whence, the equation $x^2 + ex + f = 0$, becomes

$$x^2 + \alpha x + \tfrac{1}{2}\left(q + e^2 - \frac{r}{e}\right) = 0,$$

$$\text{or, } x^2 + \alpha x + \frac{\alpha^2}{4} = \tfrac{1}{4}(\beta + \gamma)^2:$$

and $\therefore\ x = \tfrac{1}{2}\{-\alpha \pm \beta \pm \gamma\}$, two of the roots:

similarly, the equation $x^2 - ex + g = 0$, becomes

$$x^2 - \alpha x + \frac{\alpha^2}{4} = \tfrac{1}{4}(\beta - \gamma)^2,$$

which gives $x = \tfrac{1}{2}\{\alpha \pm \beta \mp \gamma\}$, the two remaining roots:

that is, the four roots of the equation are

$$\frac{-\alpha+\beta+\gamma}{2},\quad \frac{-\alpha-\beta-\gamma}{2},\quad \frac{\alpha+\beta-\gamma}{2},\quad \frac{\alpha-\beta+\gamma}{2}.$$

If instead of taking $e=\alpha$, we had assumed $e=\beta$, the four roots would have been

$$\frac{-\beta+\alpha+\gamma}{2},\quad \frac{-\beta-\alpha-\gamma}{2},\quad \frac{\beta+\alpha-\gamma}{2},\quad \frac{\beta-\alpha+\gamma}{2},$$

which are the same as the former: and hence it appears that whatever root of the reducing cubic is used, the same roots of the proposed biquadratic are obtained.

If e be the sum of two roots of the equation $x^4+qx^2+rx+s=0$, it may be shewn that e^2 has only three *different* values, without the consideration of the reducing cubic.

First, let all the roots be real and be represented by a, b, c, d: then $0=a+b+c+d$, since the coefficient of $x^3=0$: and this gives $d=-(a+b+c)$:

now, $e=$ the coefficient of the second term of the reducing quadratics $=$ the sum of two roots:

$\therefore$ all the values of e are

$$a+b,\ a+c,\ -b-c,\ b+c,\ -a-c,\ \text{and}\ -a-b:$$

$$\text{or,}\ \pm(a+b),\ \pm(a+c),\ \text{and}\ \pm(b+c):$$

whence, the values of e^2 will be

$$(a+b)^2,\ (a+c)^2,\ \text{and}\ (b+c)^2:$$

which are three in number, and being all real, cannot be determined by *Cardan's* Solution.

Secondly, let all the roots be imaginary as

$$\alpha\pm\beta\sqrt{-1},\ \text{and}\ \gamma\pm\delta\sqrt{-1}:$$

$$\text{then}\ 0=2\alpha+2\gamma,\ \text{or}\ \gamma=-\alpha,\ \text{as before}:$$

or the roots are $\alpha \pm \beta\sqrt{-1}$, and $-\alpha \pm \delta\sqrt{-1}$:
$\therefore$ the values of e are comprehended in the forms

$$\pm 2\alpha, \quad \pm(\beta + \delta)\sqrt{-1}, \text{ and } \pm(\beta - \delta)\sqrt{-1}:$$

and the values of e^2 are

$$4\alpha^2, \; -(\beta+\delta)^2, \text{ and } -(\beta-\delta)^2,$$

which are three in number, and all real, as before.

Thirdly, if the roots be two real and two imaginary, as

$$\alpha \pm \beta\sqrt{-1}, \text{ and } -\alpha \pm \delta,$$

the values of e will be

$$\pm 2\alpha, \quad \pm(\delta + \beta\sqrt{-1}), \text{ and } \pm(\delta - \beta\sqrt{-1}):$$

and the corresponding values of e^2 are

$$4\alpha^2, \; (\delta + \beta\sqrt{-1})^2, \text{ and } (\delta - \beta\sqrt{-1})^2,$$

which are three in number, one being real and two imaginary, so that *Cardan's* Solution is applicable without the irreducible case.

Ex. 27. To find the four biquadrate roots of 1 and -1.

Let $x = \sqrt[4]{1}$, and $\therefore x^4 - 1 = 0$:

but $x^4 - 1 = (x^2 - 1)(x^2 + 1) = 0$:

and this will be satisfied by making $x^2 - 1 = 0$, which gives

$$x^2 = 1 \text{ and } \therefore x = \pm 1:$$

also, by making $x^2 + 1 = 0$, from which we have $x^2 = -1$,

$$\text{and } \therefore x = \pm\sqrt{-1}:$$

so that the four roots are $1, -1, \sqrt{-1}$ and $-\sqrt{-1}$, two of which are real and two imaginary.

Again, let $y = \sqrt[4]{-1}$, and $\therefore y^4 + 1 = 0$:

$$\text{whence, } y^2 + \frac{1}{y^2} = 0, \text{ and } y^2 + 2 + \frac{1}{y^2} = 2:$$

$\therefore\ y + \dfrac{1}{y} = \pm\sqrt{2}$, and the two quadratics $y^2 \mp \sqrt{2}y + 1 = 0$, contain the four roots, which are immediately found to be

$$\frac{1 \pm \sqrt{-1}}{\sqrt{2}} \text{ and } \frac{-1 \pm \sqrt{-1}}{\sqrt{2}}.$$

Similarly, the five *fifth* roots of 1 and -1 are found by means of the equations $x^5 - 1 = 0$, and $x^5 + 1 = 0$: and so on.

Ex. 28. Given $\dfrac{x-a}{x} = \dfrac{(y-b)^2}{y^2}$ and $\dfrac{x-c}{x} = \dfrac{(y-d)^2}{y^2}$, to find the values of x and y.

From (1), $\dfrac{a}{x} = \dfrac{2b}{y} - \dfrac{b^2}{y^2}$: from (2) $\dfrac{c}{x} = \dfrac{2d}{y} - \dfrac{d^2}{y^2}$:

$$\therefore\ \frac{ad}{x} = \frac{2bd}{y} - \frac{b^2d}{y^2}, \text{ and } \frac{bc}{x} = \frac{2bd}{y} - \frac{d^2b}{y^2}:$$

$$\text{whence, } \frac{ad-bc}{x} = \frac{bd(d-b)}{y^2}, \text{ or } \frac{x}{ad-bc} = \frac{y^2}{bd(d-b)}:$$

$$\text{again, } \frac{ad^2}{x} = \frac{2bd^2}{y} - \frac{b^2d^2}{y^2}, \text{ and } \frac{b^2c}{x} = \frac{2b^2d}{y} - \frac{b^2d^2}{y^2}:$$

$$\text{whence, } \frac{ad^2 - b^2c}{x} = \frac{2bd(d-b)}{y}:$$

$$\therefore\ \frac{x}{ad-bc} \times \frac{ad^2-b^2c}{x} = \frac{y^2}{bd(d-b)} \times \frac{2bd(d-b)}{y},$$

$$\text{or } \frac{ad^2-b^2c}{ad-bc} = 2y, \text{ and } y = \frac{ad^2-b^2c}{2(ad-bc)}:$$

$$\text{also, } \frac{ad^2-b^2c}{x} \times \frac{ad^2-b^2c}{ad-bc} = \frac{2bd(d-b)}{y} \times 2y = 4bd(d-b):$$

$$\therefore\ x = \frac{(ad^2-b^2c)^2}{4bd(d-b)(ad-bc)}.$$

Ex. 29. Find all the values of x and y which satisfy the equations,

$$\frac{9a^2y^2 - 4d^4}{9bx} = 3ay - \frac{8bx + 14d^2}{9},$$

and $$\frac{2b^2x^2 + abxy - 3a^2y^2}{d^2} = 6d^2 - 6\tfrac{1}{2}bx - 13\tfrac{1}{2}ay.$$

From (1), $a^2y^2 - 3bx(ay) = \dfrac{4d^4 - 8b^2x^2 - 14bd^2x}{9}$:

and by completing the square, &c., we obtain

$$ay = \frac{8bx - 2d^2}{3}, \text{ and } ay = \frac{bx + 2d^2}{3}:$$

from (2), $a^2y^2 - \left(\dfrac{bx}{3} + \dfrac{9}{2}d^2\right) ay = \dfrac{4b^2x^2 + 13bd^2x - 12d^4}{6}$.

whence, proceeding as before, we find

$$ay = bx + 4d^2, \text{ and } ay = -\frac{4bx - 3d^2}{6}.$$

(1) $\dfrac{8bx - 2d^2}{3} = bx + 4d^2$, gives $x = \dfrac{14d^2}{5b}$.

(2) $\dfrac{8bx - 2d^2}{3} = -\dfrac{4bx - 3d^2}{6}$, gives $x = \dfrac{7d^2}{20b}$:

(3) $\dfrac{bx + 2d^2}{3} = bx + 4d^2$, gives $x = -\dfrac{5d^2}{b}$.

(4) $\dfrac{bx + 2d^2}{3} = -\dfrac{4bx - 3d^2}{6}$, gives $x = -\dfrac{d^2}{6b}$:

(5) $\dfrac{8bx - 2d^2}{3} = \dfrac{bx + 2d^2}{3}$, gives $x = \dfrac{4d^2}{7b}$:

(6) $bx + 4d^2 = -\dfrac{4bx - 3d^2}{6}$, gives $x = -\dfrac{21d^2}{10b}$.

And the simultaneous values of x and y are

$$\left.\begin{array}{l} x = \dfrac{14d^2}{5b} \\ y = \dfrac{34d^2}{5a} \end{array}\right\}, \quad \left.\begin{array}{l} x = \dfrac{7d^2}{20b} \\ y = \dfrac{4d^2}{15a} \end{array}\right\}, \quad \left.\begin{array}{l} x = -\dfrac{5d^2}{b} \\ y = -\dfrac{d^2}{a} \end{array}\right\},$$

$$\left.\begin{array}{l} x = -\dfrac{d^2}{6b} \\ y = \dfrac{11d^2}{18a} \end{array}\right\}, \quad \left.\begin{array}{l} x = \dfrac{4d^2}{7b} \\ y = \dfrac{6d^2}{7a} \end{array}\right\}, \quad \left.\begin{array}{l} x = -\dfrac{21d^2}{10b} \\ y = \dfrac{19d^2}{10a} \end{array}\right\}.$$

Ex. 30. Given $x^{x+y} = y^{4m}$, and $y^{x+y} = x^m$, to find x and y.

$$\text{From (1), } x = y^{\frac{4m}{x+y}}: \text{ from (2), } x = y^{\frac{x+y}{m}}:$$

$$\therefore\ y^{\frac{4m}{x+y}} = y^{\frac{x+y}{m}}, \text{ and } \frac{x+y}{m} = \frac{4m}{x+y}:$$

whence, $(x+y)^2 = 4m^2$, and $\therefore\ x + y = \pm 2m$:

(1), we have $x = y^{\frac{x+y}{m}} = y^2$, and $\therefore\ y^2 + y = 2m$,
which gives

$$y = \tfrac{1}{2}\{-1 \pm \sqrt{8m+1}\}, \text{ and } x = \tfrac{1}{2}\{4m + 1 \mp \sqrt{8m+1}\}:$$

(2), we have $x = \dfrac{1}{y^2}$, and $\therefore\ y^3 + 2my^2 + 1 = 0$,

which may be solved by *Cardan's* Rule, when possible.

Ex. 31. Given the n following equations, viz.:

$$a_1x_1 + a_2x_2 + a_3x_3 + \&c. + a_nx_n = 0,$$
$$b_1x_1 + b_2x_2 + b_3x_3 + \&c. + b_nx_n = 0,$$
$$c_1x_1 + c_2x_2 + c_3x_3 + \&c. + c_nx_n = 0, \ \&c.,$$
$$k_1x_1^m + k_2x_2^m + k_3x_3^m + \&c. + k_nx_n^m = k:$$

to find the values of x_1, x_2, x_3, &c., x_n.

Here, by means of the $n-1$ former equations, each of the quantities x_2, x_3, &c., x_n may be expressed in terms of x_1: whence, if

$$x_2 = l_2 x_1,\ x_3 = l_3 x_1,\ \&c.,\ x_n = l_n x_1,$$

we shall have by substitution in the last

$$k_1 x_1^m + k_2 l_2^m x_1^m + k_3 l_3^m x_1^m + \&c. + k_n l_n^m x_1^m = k:$$

and therefore $x_1 = \left(\dfrac{k}{k_1 + k_2 l_2^m + k_3 l_3^m + \&c. + k_n l_n^m}\right)^{\frac{1}{m}}$:

from which the values of x_2, x_3, &c., x_n are immediately derived.

Ex. 32. Given the n equations following, viz.:

$$a_1 x_1 + a_2 x_2 + a_3 x_3 + \&c. + a_n x_n = A,$$

$$b_1 x_1 + b_2 x_2 + b_3 x_3 + \&c. + b_n x_n = B,$$

$$c_1 x_1 + c_2 x_2 + c_3 x_3 + \&c. + c_n x_n = C,\ \&c.:$$

to find the values of x_1, x_2, x_3, &c., x_n.

Multiplying the members of the first, second, third, &c. equations by 1, p, q, &c. respectively, we have

$$a_1 x_1 + a_2 x_2 + a_3 x_3 + \&c. + a_n x_n = A,$$

$$p b_1 x_1 + p b_2 x_2 + p b_3 x_3 + \&c. + p b_n x_n = pB,$$

$$q c_1 x_1 + q c_2 x_2 + q c_3 x_3 + \&c. + q c_n x_n = qC,\ \&c.:$$

and by addition in vertical columns, we find

$$(a_1 + pb_1 + qc_1 + \&c.)\, x_1 + (a_2 + pb_2 + qc_2 + \&c.)\, x_2$$
$$+ (a_3 + pb_3 + qc_3 + \&c.)\, x_3 + \&c. + (a_n + pb_n + qc_n + \&c.)\, x_n$$
$$= A + pB + qC + \&c.:$$

whence, if the coefficients of x_2, x_3, &c., x_n in this equation, be assumed $= 0$, we shall have

$$x_1 = \frac{A + pB + qC + \&c.}{a_1 + pb_1 + qc_1 + \&c.},$$

where the values of p, q, &c. may evidently be determined by means of the $n-1$ equations,

$$a_2 + b_2 p + c_2 q + \&c. = 0,$$

$$a_3 + b_3 p + c_3 q + \&c. = 0, \ \&c. = \&c.$$

$$a_n + b_n p + c_n q + \&c. = 0,$$

as pointed out in the last example: and similarly of x_2, x_3, &c., x_n.

Ex. 33. Let there be given the n following equations:

$$a_1 x_1 x_2 + b_1 x_1 + c_1 x_2 + d_1 = 0,$$

$$a_2 x_2 x_3 + b_2 x_2 + c_2 x_3 + d_2 = 0,$$

$$a_3 x_3 x_4 + b_3 x_3 + c_3 x_4 + d_3 = 0, \ \&c.$$

$$a_n x_n x_1 + b_n x_n + c_n x_1 + d_n = 0:$$

to find the values of x_1, x_2, x_3, &c., x_n.

From (1), we have $x_2 = -\dfrac{b_1 x_1 + d_1}{a_1 x_1 + c_1}$:

whence we shall find x_3 in terms of x_1 by means of (2): and continuing the process, we shall at length obtain x_n in the form

$$\frac{Ax_1 + B}{Cx_1 + D}:$$

which being substituted in the last equation, gives

$$a_n \left(\frac{Ax_1^2 + Bx_1}{Cx_1 + D}\right) + b_n \left(\frac{Ax_1 + B}{Cx_1 + D}\right) + c_n x_1 + d_n = 0,$$

a quadratic equation, from which x_1 may be determined: and thence x_2, x_3, &c., x_n will become known.

ELIMINATION.

25. The following examples, in addition to what has been said in the text, will further illustrate this subject.

Ex. 1. Eliminate x by means of the equations

$$x^3 + y^3 - (2y^2 + 1)(x + 1) = 0, \text{ and } x^2 - y^2 - xy - 1 = 0.$$

From (2), $x^3 - xy^2 - x^2 y - x = 0$:

from (1), $x^3 + y^3 - (2y^2 + 1)(x + 1) = 0$:

$$\therefore\ y^3 + x^2y - xy^2 - 2y^2 - 1 = 0:$$

but $x^2y - xy^2 = y^3 + y$, from (2):

$$\therefore\ 2y^3 - 2y^2 + y - 1 = 0, \text{ or } (2y^2 + 1)(y - 1) = 0,$$

an equation involving only y.

Ex. 2. Given the three following equations

$$x + y + z = a,\ xy + xz + yz = b^2, \text{ and } xyz = c^3,$$

to eliminate y and z.

$$\text{From (1), } x^3 + x^2y + x^2z = ax^2:$$

$$\text{from (2), } x^2y + x^2z + xyz = b^2x:$$

$$\text{and therefore, } x^3 - xyz = ax^2 - b^2x:$$

$$\text{from (3), } \quad xyz = c^3:$$

$$\therefore\ x^3 = ax^2 - b^2x + c^3, \text{ or } x^3 - ax^2 + b^2x - c^3 = 0,$$

a cubic equation involving only one unknown symbol x.

Ex. 3. By means of the three equations $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$,

$\alpha = \dfrac{a^2 - b^2}{a^4}x^3$, and $\beta = \dfrac{a^2 - b^2}{b^4}y^3$, eliminate x and y.

$$\text{From (2), } a\alpha = (a^2 - b^2)\left(\frac{x}{a}\right)^3, \text{ and } \therefore\ (a\alpha)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}\left(\frac{x}{a}\right)^2:$$

$$\text{from (3), } b\beta = (a^2 - b^2)\left(\frac{y}{b}\right)^3, \text{ and } \therefore\ (b\beta)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}\left(\frac{y}{b}\right)^2:$$

$$\text{whence, } (a\alpha)^{\frac{2}{3}} + (b\beta)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = (a^2 - b^2)^{\frac{2}{3}},$$

$$\text{since } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ by the first equation.}$$

Ex. 4. Given $xy = a^2$, $2\alpha x^3 = a^4 + 3x^4$, and $2\beta y^3 = a^4 + 3y^4$, to eliminate x and y.

Here, $\beta + a = \frac{1}{2x^3y^3}\{a^4(x^3+y^3) + 3x^3y^3(x+y)\}$

$$= \frac{1}{2a^2}(x+y)^3, \text{ since } xy = a^2:$$

also, $\beta - a = \frac{1}{2a^2}(x-y)^3$, by a similar process:

$$\therefore\ (\beta + a)^{\frac{1}{3}} + (\beta - a)^{\frac{1}{3}} = \frac{2x}{\sqrt[3]{2a^2}},$$

$$\text{and }\ (\beta + a)^{\frac{1}{3}} - (\beta - a)^{\frac{1}{3}} = \frac{2y}{\sqrt[3]{2a^2}}:$$

$$\text{whence, }\ (\beta + a)^{\frac{2}{3}} - (\beta - a)^{\frac{2}{3}} = \frac{4xy}{\sqrt[3]{4a^4}} = (4a)^{\frac{2}{3}},$$

an equation which does not contain x or y.

Ex. 5. Find the relation of the coefficients of

$$x^3 - px^2 + qx - r = 0,$$

when the roots are in geometrical progression.

Let $\frac{a}{\beta}$, a and $a\beta$ be the roots: then we have

$$\frac{a}{\beta} + a + a\beta = p,\ \frac{a^2}{\beta} + a^2 + a^2\beta = q \text{ and } a^3 = r,$$

to eliminate a and β.

From (1) and (2), we have $pa = q$, and from (3), $a = r^{\frac{1}{3}}$: $\therefore\ pr^{\frac{1}{3}} = q$, or $p^3r = q^3$ is the relation required: and this condition we see fulfilled in the equation $x^3 - 7x^2 + 14x - 8 = 0$, which has been previously solved.

Similarly, when the roots are in arithmetical or harmonical progression.

INEQUALITIES.

26. Questions involving inequalities are always treated according to the principle laid down in article (49).

Ex. 1. Prove that $2x^4+1$ is greater than $2x^3+x^2$, whatever positive integral value be assigned to x.

Here, $2x^4+1$ is $>$ or $< 2x^3+x^2$,

$$\text{according as } 2x^4-2x^3 \text{ is } > \text{ or } < x^2-1,$$

$$\text{as } 2x^3(x-1) \text{ is } > \text{ or } < (x+1)(x-1),$$

$$\text{as } 2x^3 \text{ is } > \text{ or } < x+1,$$

$$\text{as } 2(x^3-1) \text{ is } > \text{ or } < x-1,$$

$$\text{as } 2(x-1)(x^2+x+1) \text{ is } > \text{ or } < x-1,$$

$$\text{as } 2(x^2+x+1) \text{ is } > \text{ or } < 1,$$

$$\text{as } (4x^2+4x+1)+1 \text{ is } > \text{ or } < 0:$$

but $(4x^2+4x+1)+1=(2x+1)^2+1^2$ being the sum of two squares is always greater than 0,

$$\text{and } \therefore\ 2x^4+1 \text{ is } > 2x^3+x^2.$$

In consequence of the equal divisions by $x-1$, which has been treated as an arithmetical quantity, this conclusion will not hold good when $x=1$: and accordingly we find that the two quantities are equal for this particular of x, and for no other.

Ex. 2. Compare the binomial surd quantities

$$\sqrt{5}+\sqrt{14} \text{ and } \sqrt{3}+3\sqrt{2},$$

considered arithmetically.

Here, $\sqrt{5}+\sqrt{14}$ is $>$ or $<\sqrt{3}+3\sqrt{2}$,

$$\text{according as } 19+2\sqrt{70} \text{ is } > \text{ or } < 21+6\sqrt{6},$$

$$\text{as } \sqrt{70} \text{ is } > \text{ or } < 1+3\sqrt{6},$$

$$\text{as } 70 \text{ is } > \text{ or } < 55+6\sqrt{6},$$

as 15 is $>$ or $< 6\sqrt{6}$, as 225 is $>$ or < 216:

that is, $\sqrt{5}+\sqrt{14}$ is greater than $\sqrt{3}+3\sqrt{2}$.

Ex. 3. Prove that $\sqrt{a^2+b^2}+\dfrac{ab}{\sqrt{a^2+b^2}}$ is greater than $a+b$, whatever be the unequal values of a and b.

Here, $\sqrt{a^2+b^2}+\dfrac{ab}{\sqrt{a^2+b^2}}$ is $>$ or $< a+b$,

according as a^2+b^2+ab is $>$ or $< (a+b)\sqrt{a^2+b^2}$,

as a^3-b^3 is $>$ or $< (a^2-b^2)\sqrt{a^2+b^2}$,

as $a^6-2a^3b^3+b^6$ is $>$ or $< a^6-a^4b^2-a^2b^4+b^6$,

as $a^2b^2(a^2-2ab+b^2)$ is $>$ or < 0,

as $(ab)^2(a-b)^2$ is $>$ or < 0:

that is, $\sqrt{a^2+b^2}+\dfrac{ab}{\sqrt{a^2+b^2}}$ is always greater than $a+b$.

Ex. 4. What is the integer value of x,

when $\frac{1}{4}(x+2)+\frac{1}{3}x$ is less than $\frac{1}{2}(x-4)+3$,

and greater than $\frac{1}{2}(x+1)+\frac{1}{3}$?

Multiplying all the quantities by 12, we have

$$7x+6<6x+12 \text{ and } >6x+10:$$

$$\therefore\ x \text{ is } <6 \text{ and } >4, \text{ or } x=5.$$

Ex. 5. Resolve a into two factors, whose sum shall be the least possible.

Let the factors be x and $\dfrac{a}{x}$, and assume $x+\dfrac{a}{x}=m$:

which gives $x=\frac{1}{2}(m\pm\sqrt{m^2-4a})$:

therefore, in order that the factors may be real magnitudes, m^2 must not be less than $4a$, and for the extreme case in which they are real, we have

$$x=\tfrac{1}{2}m=\sqrt{a}, \text{ and } \therefore\ \frac{a}{x}=\sqrt{a},$$

so that, the required factors are equal to each other, and their sum $= 2\sqrt{a}$.

The same result may be obtained from the following consideration.

Let one of the factors $x = c\sqrt{a}$, and $\therefore\ \frac{a}{x} = \frac{\sqrt{a}}{c}$:

whence $x + \frac{a}{x} = \left(c + \frac{1}{c}\right)\sqrt{a}$, which will be greater than $2\sqrt{a}$, unless $c = 1$, as appears from article (81).

MISCELLANEOUS PROBLEMS.

27. The following small collection of Problems, dependent upon the principles explained in the text, will afford additional hints for the acquirement of dexterity in the use of Algebra.

Prob. 1. A garrison which had provisions for 30 months, was doubled at the end of 4 months, and increased by 400 men 3 months afterwards, and the provisions were then exhausted in fifteen months from the first: find the original number of men.

Let x denote the original number of men, and a the quantity of provisions consumed by 1 man in 1 month:

$$\therefore\ 30ax = \text{the whole quantity of provisions:}$$

also, $4ax$ = quantity of provisions consumed in first 4 months:

$2x \times 3 \times a = 6ax$ .... next 3 months:

and $(2x + 400) \times 8a = 16ax + 3200a$ = ... next 8 months:

$$\text{whence, } 30ax = 26ax + 3200a, \text{ and } \therefore\ x = 800 \text{ men.}$$

Prob. 2. A gentleman bequeaths his property as follows: to his eldest child he leaves £1800. and one-sixth of the rest of his property: to the second twice that sum, and one-sixth of what then remained: to the third three times the same sum, and one-sixth of the remainder, and so on: and by this arrange-

ment his property is divided equally among his children: how many were there, and what was their fortune?

If x = the whole property, then we shall have

$$1800 + \frac{x-1800}{6} = 1500 + \frac{x}{6} = \text{the fortune of the eldest:}$$

$$\text{also, } x - 1500 - \frac{x}{6} = \frac{5x}{6} - 1500 = \text{the sum remaining:}$$

$$\therefore\ 2750 + \frac{5x}{36} = \text{the fortune of the second:}$$

$$\text{whence, } 1500 + \frac{x}{6} = 2750 + \frac{5x}{36}, \text{ and } \therefore\ x = \pounds 45000:$$

$$\text{also, } 1500 + \frac{x}{6} = \pounds 9000., \text{ the fortune of each:}$$

$$\text{and } \therefore\ \frac{45000}{9000} = 5, \text{ the number of children.}$$

This problem will furnish the student with a good exercise in tracing the possibility of the conditions proposed, by finding the general relation between the fortunes of any two consecutive children.

PROB. 3. A pack of p cards is distributed into n heaps, so that the number of pips on the lowest cards, together with the number of cards laid upon them, is the same given number m for each heap, and the number of cards then remaining is found to be r: required the number of pips on all the lowest cards.

Let x = the number of pips required:
then, since mn = the number of pips, together with the numbers of cards laid upon the lowest, $mn - x$ = the number of cards laid upon all the lowest:

$$\therefore\ mn - x + n = \text{the number of cards in all the heaps:}$$

$$\text{whence, } mn - x + n + r = p, \text{ and } x = (m+1)n + r - p.$$

This trick may readily be performed by means of a common pack of cards.

PROB. 4. A and B travelled on the same road, and at the same rate from H to L. At the 50th milestone from L, A overtook a flock of geese, which were proceeding at the rate of 3 miles in 2 hours; and 2 hours afterwards he met a stage-waggon which was moving at the rate of 9 miles in 4 hours. B overtook the flock of geese at the 45th milestone, and met the stage-waggon 40 minutes before he came to the 31st milestone: where was B when A reached L?

Let $x =$ the rate of travelling of A and B:

$\therefore$ B approaches the waggon $x + \frac{9}{4}$ miles per hour:

and $\frac{3}{2} : 5 :: 1 : \frac{10}{3} =$ the number of hours in which B overtook the geese after A:

whence, $\frac{10x}{3} =$ space he passed over in that time, and therefore $\frac{10x}{3} - 5 = B$'s distance from A.

Again, A met the waggon $50 - 2x$ miles from L, and B met it $31 + \frac{2x}{3}$ miles from L:

$\therefore$ it had travelled $\frac{8x}{3} - 19$ miles in that time:

whence, the time elapsed between A and B meeting the waggon is $\frac{4}{9}\left(\frac{8x}{3} - 19\right)$:

and A's distance from $B = \frac{4}{9}\left(\frac{8x}{3} - 19\right) \times \left(x + \frac{9}{4}\right)$:

$$\therefore \left(\frac{8x}{3} - 19\right)\left(\frac{4x}{9} + 1\right) = \frac{10x}{3} - 5, \text{ by the question,}$$

which gives $x = 9$, the rate of travelling of A and B:

$$\text{and } \therefore \frac{10x}{3} - 5 = 25 \text{ miles, the distance required.}$$

Prob. 5. To find two magnitudes whose product is a, and the difference of whose squares is $2b$.

Here, $xy = a$, or $2\sqrt{-1}\,xy = 2a\sqrt{-1}$, and $x^2 - y^2 = 2b$:

$$\therefore\ x^2 + 2\sqrt{-1}\,xy - y^2 = 2b + 2a\sqrt{-1},$$

$$x^2 - 2\sqrt{-1}\,xy - y^2 = 2b - 2a\sqrt{-1}:$$

$$\text{whence, } x + y\sqrt{-1} = (2b + 2a\sqrt{-1})^{\frac{1}{2}},$$

$$\text{and } x - y\sqrt{-1} = (2b - 2a\sqrt{-1})^{\frac{1}{2}}:$$

$$\therefore\ x = \tfrac{1}{2}\left\{(2b + 2a\sqrt{-1})^{\frac{1}{2}} + (2b - 2a\sqrt{-1})^{\frac{1}{2}}\right\},$$

$$\text{and } y = \frac{1}{2\sqrt{-1}}\left\{(2b + 2a\sqrt{-1})^{\frac{1}{2}} - (2b - 2a\sqrt{-1})^{\frac{1}{2}}\right\}.$$

These values of x and y, though expressed in symbolical forms, are both real magnitudes by article (263), but being *irreducible*, will not be convenient solutions in their present forms:

if however, $\dfrac{x}{y} = \dfrac{b + \sqrt{a^2 + b^2}}{a}$ be combined with $xy = a$,

we have immediately

$$x = \pm\left\{b + \sqrt{a^2 + b^2}\right\}^{\frac{1}{2}}, \text{ and } y = \pm\frac{a}{\left\{b + \sqrt{a^2 + b^2}\right\}^{\frac{1}{2}}},$$

which are the quantities required.

Prob. 6. A waterman rows a given distance a and back again in b hours, and finds that he can row c miles with the tide for d miles against it: required the rate of the tide, and the rate of rowing, and also the times of rowing down and up the stream.

Let x = the time with the tide, and $\therefore b - x$ = the time against it: $\therefore \dfrac{a}{x}$ and $\dfrac{a}{b-x}$ are the rates with and against

the tide: whence, $\frac{a}{x} : \frac{a}{b-x} = c : d$ by the question, which gives $x = \frac{bd}{c+d}$, and $\therefore\ b - x = \frac{bc}{c+d}$.

Also, $\frac{a(c+d)}{bd}$ = the rate down the stream

= the rate of rowing + the rate of the tide:

and $\frac{a(c+d)}{bc}$ = the rate up the stream

= the rate of rowing − the rate of the tide:

$\therefore$ rate of tide $= \frac{a(c^2-d^2)}{2bcd}$: rate of rowing $= \frac{a(c+d)^2}{2bcd}$.

Prob. 7. If from a cask of wine containing a gallons, b gallons be drawn off and the vessel filled up with water, and this be repeated n times successively: find the quantity of wine then remaining.

Let a_1, a_2, a_3, &c., a_n denote the quantities of wine remaining after the operation has been repeated *once*, *twice*, *thrice*, &c., n times respectively: then it is manifest that

$$a : a_1 = a : a - b:$$

but since the strength of the mixture, and therefore the wine in it decreases at every operation in the ratio of $a : a - b$, we have

$$a_1 : a_2 = a : a - b,$$
$$a_2 : a_3 = a : a - b, \text{ \&c.}$$
$$a_{n-1} : a_n = a : a - b:$$

whence, $a : a_n = a^n : (a-b)^n$:

and $\therefore\ a_n = \frac{(a-b)^n}{a^{n-1}}$, the quantity required.

If n be a very large number, we shall have

$$a_n = a\left(1 - \frac{nb}{a} + \frac{n^2b^2}{1.2a^2} - \frac{n^3b^3}{1.2.3a^3} + \text{\&c.}\right) = ae^{-\frac{nb}{a}},$$

as appears from the expansion of e^{-x} hereafter given.

PROB. 8. There are three casks, each of which contains a mixture of water, wine and brandy in given ratios: what quantity must be taken from each to form a fourth mixture which shall contain a given quantity of the said ingredients?

Let a_1, b_1, c_1 denote the water, wine and brandy in the 1st cask:

a_2, b_2, c_2 .... 2nd ...

a_3, b_3, c_3 .... 3rd ...

a_4, b_4, c_4 .... 4th ...

and assume x, y, z to represent the whole quantities taken from the first, second, and third casks to form the fourth:

$$\text{then, } a_1 + b_1 + c_1 : x :: a_1 : \frac{a_1 x}{a_1 + b_1 + c_1} = \text{water from 1st:}$$

$$a_2 + b_2 + c_2 : y :: a_2 : \frac{a_2 y}{a_2 + b_2 + c_2} = \ldots\ldots\ldots\ldots \text{ 2nd:}$$

$$a_3 + b_3 + c_3 : z :: a_3 : \frac{a_3 z}{a_3 + b_3 + c_3} = \ldots\ldots\ldots\ldots \text{ 3rd:}$$

$$\text{whence, } \frac{a_1 x}{a_1 + b_1 + c_1} + \frac{a_2 y}{a_2 + b_2 + c_2} + \frac{a_3 z}{a_3 + b_3 + c_3} = a_4.$$

Similarly, two additional equations may be obtained, and thus the values of x, y, z will be determined.

PROB. 9. If a oxen in m weeks eat b acres of grass, and c oxen eat d acres in n weeks: how many oxen will eat e acres in p weeks, the grass being supposed to grow uniformly?

Let x = the number of oxen required, α = the grass upon an acre at first, and β = the increase of grass upon an acre in a week:

$\therefore\ \alpha + m\beta$ = the grass on 1 acre in m weeks:

$\alpha + n\beta$ = ... n ......

$\alpha + p\beta$ = ... p ......

$\therefore\ b(a + m\beta) =$ the grass on b acres in m weeks:

$d(a + n\beta) =$ .... d .... n ......

$e(a + p\beta) =$ .... e .... p ......

now, it is manifest that the quantity of grass consumed will vary as the number of oxen and time jointly: whence, we have

$$b(a + m\beta) : d(a + n\beta) = ma : nc:$$

$$b(a + m\beta) : e(a + p\beta) = ma : px:$$

$$\therefore \text{ from (1), we find } \beta = \frac{(mad - nbc)a}{mn(bc - ad)}:$$

$$\text{and from (2), we have } x = \frac{mae(a + p\beta)}{pb(a + m\beta)}$$

$$= \left(\frac{m-p}{m-n}\right)\frac{nce}{pd} - \left(\frac{n-p}{m-n}\right)\frac{mae}{pb}, \text{ the number required.}$$

Hence, $(m - n)\dfrac{px}{e} = (p - n)\dfrac{ma}{b} - (p - m)\dfrac{nc}{d}$, is an useful practical theorem, from which any one of the quantities concerned may be determined, when the rest are given.

PROB. 10. A debt of a£ accumulating at compound interest, is discharged in n years by annual payments of $\dfrac{a}{m}$£: prove that $n = -\dfrac{\log(1 - mr)}{\log(1 + r)}$, where r is the interest of 1£ for 1 year.

By the formula, $PR^n = \left(\dfrac{R^n - 1}{R - 1}\right)A$, if we make $P = a$, $R = 1 + r$, and $A = \dfrac{a}{m}$, we shall have

$$(1 + r)^n = \frac{1}{1 - mr}, \text{ and } \therefore\ n = -\frac{\log(1 - mr)}{\log(1 + r)}.$$

PROB. 11. A person spends in the first year m times the interest of his property: in the second year, $2m$ times

that of the remainder: in the third year, $3m$ times that at the end of the second, and so on: and at the end of $2p$ years he has nothing left: shew that in the p^{th} year he spends as much as he has left at the end of that year.

Let P_x = his property at the end of the x^{th} year:

$$\therefore \text{ the interest in the } (x+1)^{\text{th}} \text{ year} = rP_x:$$

and his expenditure in this year $= (x+1)\,mrP_x$:

$$\therefore \text{ the property left} = P_x(1+r) - (x+1)\,mrP_x:$$

whence putting $2p-1$ for x, we have

$$\{1+r-2pmr\}\,P_{2p-1} = 0, \text{ and } \therefore\ 1+r = 2pmr:$$

similarly, his expenditure in the p^{th} year is $pmrP_{p-1}$, and his property left at the end of that year

$$= \{1+r-pmr\}\,P_{p-1}$$

$$= (2pmr - pmr)\,P_{p-1} = pmr\,P_{p-1}.$$

PROB. 12. A mortgage is taken on an estate worth n acres of it: but land rises p per cent. in price, and in consequence, the mortgage is worth only n_1 acres, and it is then paid off. During the continuance of high prices, another mortgage is taken worth n acres as before: but prices returning to their former level, the mortgage is worth n_2 acres: shew that

$$n - n_1 : n_2 - n = 1 : 1 + \frac{p}{100}.$$

Let a = first price of an acre, $\therefore\ a + \dfrac{pa}{100}$ = second price:

$$\therefore \text{ the amount of the first mortgage} = na = n_1\left(a + \frac{pa}{100}\right):$$

$$\text{also, } n\left(a + \frac{pa}{100}\right) = n_2 a, \text{ by the question}:$$

$$\text{whence, } n_2 - n = \frac{p}{100}\,n = \frac{p}{100}\left(1 + \frac{p}{100}\right)n_1 = \left(1 + \frac{p}{100}\right)(n - n_1):$$

$$\therefore\ n - n_1 : n_2 - n = 1 : 1 + \frac{p}{100}.$$

This result shews that the advantage to the mortgager from a rise in the price of land, is less than the disadvantage to the mortgagee from a fall in price, in the ratio of

$$1 : 1 + \frac{p}{100}.$$

PROB. 13. If P denote the population of a country at a given period, $\frac{1}{p}$ the ratio of mortality, $\frac{1}{q}$ that of births: and if A represent the amount of population in n years, then will

$$A = P\left\{1 + \frac{p-q}{pq}\right\}^n.$$

For, the rate of increase $= \frac{1}{q} - \frac{1}{p} = \frac{p-q}{pq}$:

$$\therefore\ 1 : 1 + \frac{p-q}{pq} = P : P\left\{1 + \frac{p-q}{pq}\right\},$$

the population at the end of one year:

$$\text{similarly, } 1 : 1 + \frac{p-q}{pq} = P\left\{1 + \frac{p-q}{pq}\right\} : P\left\{1 + \frac{p-q}{pq}\right\}^2,$$

the population at the end of two years: &c.:

whence, at the end of n years, we shall have

$$A = P\left\{1 + \frac{p-q}{pq}\right\}^n:$$

from which, if four of the involved quantities be given, the remaining one may be found.

PROB. 14. If the number of persons born in any year be $\frac{1}{45}$th of the whole population at the commencement of that year, and the number of those who die be $\frac{1}{60}$th of it: find in how many years the population will be doubled, given $\log 2 = .301030$, $\log 180 = 2.255272$ and $\log 181 = 2.257679$.

Here, $p = 60$, $q = 45$, and $\therefore \dfrac{p-q}{pq} = \dfrac{15}{60 \times 45} = \dfrac{1}{180}$:

$$\text{whence,} \quad 2P = P\left\{\frac{181}{180}\right\}^n, \text{ by the question:}$$

$$\therefore n = \frac{\log 2}{\log 181 - \log 180} = \frac{.301030}{.002407} = 125 \text{ years, nearly.}$$

PROB. 15. A gives to B as many counters as he has already, and B returns to A as many as he had then kept: and they afterwards find their numbers to be a and b respectively: what number had each at first?

Let $x = A$'s number, $\therefore a + b - x = B$'s number at first:

$\therefore 2x - a - b = A$'s number after his present to B:

and $2(a + b - x) = B$'s number then:

$\therefore 4x - 2(a + b) = A$'s number after B's present:

whence, $4x - 2(a + b) = a$, by the question:

$$\therefore x = \frac{3a + 2b}{4}, \quad \text{and} \quad a + b - x = \frac{a + 2b}{4},$$

the original numbers of A and B.

If the same process be repeated a second time, we have merely to substitute $\dfrac{3a+2b}{4}$ and $\dfrac{a+2b}{4}$ in the places of a and b, and thus, we find

$$A\text{'s original number} = \frac{11a + 10b}{16}, \quad \text{and } B\text{'s} = \frac{5a + 6b}{16}.$$

Also, for a third time, we put $\dfrac{11a + 10b}{16}$ and $\dfrac{5a+6b}{16}$ in the places of a and b, and thus have

$$A\text{'s original number} = \frac{43a + 42b}{64}, \quad \text{and } B\text{'s} = \frac{21a + 22b}{64}.$$

And for a fourth time, we shall have likewise

$$A\text{'s original number} = \frac{171a + 170b}{256}, \quad B\text{'s} = \frac{85a + 86b}{256} : \&c.:$$

but in the numerators of the original numbers of A, we observe that the coefficients 3, 11, 43, 171, &c. of a are so connected that each of them is less by 1, than four times that which immediately precedes it, and that the coefficients of b are the doubles of the numbers 1, 5, 21, 85, &c.: also,

$3 = 4 - 1:$	$1 = 1:$
$11 = 4^2 - 4 - 1:$	$5 = 4 + 1:$
$43 = 4^3 - 4^2 - 4 - 1:$	$21 = 4^2 + 4 + 1:$
$171 = 4^4 - 4^3 - 4^2 - 4 - 1:$	$85 = 4^3 + 4^2 + 4 + 1:$
&c.	&c.

from which, by induction, we determine the original numbers of A and B to be of the forms

$$\frac{2 \cdot 4^n + 1}{3 \cdot 4^n} a + \frac{2(4^n - 1)}{3 \cdot 4^n} b,$$

$$\frac{4^n - 1}{3 \cdot 4^n} a + \frac{4^n + 2}{3 \cdot 4^n} b,$$

when the prescribed exchange has been effected n times.*

THE RULES OF POSITION.

28. Def. In *Single Position* are considered those questions, wherein the result of the operations upon the unknown quantity is always some multiple, part or parts of the quantity itself: and the applicability of the rule is entirely directed by this circumstance.

Let x be a number required, which is to undergo such operations that the result is a: also, let s be a *supposed* number, which by the same operations gives a result b:

$$\text{then, } \frac{x}{a} = \frac{s}{b}, \text{ and } \therefore\ x = \frac{a}{b} s:$$

and this expression put into words at length furnishes the rule commonly employed in questions of this description.

Ex. Find a number, which being increased by its fourth and seventh parts, becomes 39.

Suppose the number to be 24: then, the sum proposed $= 24 + 6 + 3\frac{3}{7} = 33\frac{3}{7}$, whereas it ought to have been 39:

$$\text{whence, } x = \frac{39}{33\frac{3}{7}} \times 24 = \frac{7}{6} \times 24 = 28.$$

29. Def. In *Double Position*, the result is no longer a certain multiple, part or parts of the quantity itself, but involves further the addition or subtraction of some given magnitude: and the rule will be useless unless this is the case.

Let x be required so as to satisfy $ax + b = c$: and suppose s and s' when put for x, not to satisfy the condition, but to give the errors e and e' both in excess: then, we have

$$as + b = c + e, \text{ and } as' + b = c + e':$$

$$\therefore\ a(s - x) = e, \text{ and } a(s' - x) = e':$$

$$\text{whence, } \frac{s - x}{s' - x} = \frac{e}{e'}, \text{ and } \therefore\ x = \frac{es' - e's}{e - e'},$$

which expressed in words is the rule.

If the errors be both in defect, the result is the same: but if one of the errors be in excess and the other in defect,

$$x = \frac{es' + e's}{e + e'}.$$

Ex. What number is that which being divided by 9, and the quotient diminished by 3, three times the remainder shall be 30?

Let $144 = s$, then the result $= 39$, and $e = 9$:

let $126 = s'$, then the result $= 33$, and $e' = 3$:

$$\text{whence, the number required} = \frac{es' - e's}{e - e'} = 117.$$

RATIO AND PROPORTION.

30. To find the approximate value of $(a + x)^m : a^m$, when x is very small compared to a.

Here,

$$\frac{(a+x)^m}{a^m} = \left(1 + \frac{x}{a}\right)^m = 1 + m\left(\frac{x}{a}\right) + \frac{m(m-1)}{1.2}\left(\frac{x}{a}\right)^2 + \&c.:$$

$$\therefore \text{ the first approximation} = 1 + m\left(\frac{x}{a}\right) = \frac{a + mx}{a}:$$

$$\text{the second approximation} = 1 + m\left(\frac{x}{a}\right) + \frac{m(m-1)}{1.2}\left(\frac{x}{a}\right)^2: \&c.$$

$$\text{Similarly, } \frac{(a+x)^{\frac{1}{m}}}{a^{\frac{1}{m}}} = 1 + \frac{1}{m}\left(\frac{x}{a}\right) + \frac{1(1-m)}{1.2.m^2}\left(\frac{x}{a}\right)^2 + \&c.,$$

from which the approximations may be found as before.

31. Let a and b denote two incommensurable magnitudes which admit of no common measure whatever; and suppose $b = nx$, and a to lie between mx and $(m + 1)x$: then we shall have

$$\frac{a}{b} > \frac{m}{n} \text{ but } < \frac{m+1}{n}, \text{ or } \frac{a}{b} - \frac{m}{n} < \frac{1}{n}:$$

and this, by the diminution of x and consequent increase of n, may obviously be made less than any assignable quantity: therefore, whatever is proved of the ratio $m : n$ in this case, holds good of the ratio $a : b$.

Again, if $\frac{a}{b}$ and $\frac{c}{d}$ represent two incommensurable ratios, which can both be proved to lie between $\frac{m}{n}$ and $\frac{m+1}{n}$, whatever be the magnitudes of m and n, we shall have

$$\frac{a}{b} - \frac{c}{d} < \frac{1}{n}:$$

and therefore, by reasoning as before, we have $\frac{a}{b} = \frac{c}{d}$, or the incommensurable ratio $a : b$ is equal to the incommensurable ratio $c : d$: and thus proportionality is established among the incommensurable magnitudes a, b, c and d.

PROGRESSIONS.

32. To find the sums of the powers of the terms of an arithmetical progression.

Proceeding exactly as in (3) of article (217), we shall find s_{m-1}

$$=\frac{(a+nd)^m - a^m}{md} - (m-1)s_{m-2}\frac{d}{1.2} - (m-1)(m-2)s_{m-3}\frac{d^2}{1.2.3} - \&c.:$$

and if m be put for $m-1$ in both members, the result will be rendered more convenient for practice.

33. In a geometrical progression, we have seen that $l = ar^{n-1}$, and $s = \frac{a(r^n - 1)}{r - 1}$:

whence, if $ar - a = a(r-1)$ be very small, we have

$$r - 1 = \frac{ar - a}{a} = \frac{d}{a}, \text{ suppose}:$$

$$\therefore\ l = a\left\{1 + \frac{d}{a}\right\}^{n-1} = a + (n-1)d, \text{ nearly}:$$

$$\text{also, } s = a\left\{\frac{\left(1+\frac{d}{a}\right)^n - 1}{\frac{d}{a}}\right\} = \{2a + (n-1)d\}\frac{n}{2}, \text{ nearly}:$$

and these are the expressions already investigated for an arithmetical progression: from which we infer that quantities in geometrical progression may ultimately be regarded as forming an arithmetical progression: and conversely.

34. Though the sum of a series of quantities in harmonical progression cannot be generally exhibited, there are some cases in which a good approximation may be found.

Thus, if b be very small compared with a, we have

$$a + 2b = \frac{(a+b)^2}{a}, \quad a + 3b = \frac{(a+b)^3}{a^2}, \quad \text{\&c. nearly:}$$

$$\therefore \frac{1}{a+b} + \frac{1}{a+2b} + \frac{1}{a+3b} + \text{\&c. to } n \text{ terms} = \frac{(a+b)^n - a^n}{b(a+b)^n}.$$

From this combined with the results of the last article, it is inferred that all the progressions may be regarded as coincident, whenever the difference of any two consecutive terms vanishes with respect to the terms themselves.

35. If three equidistant terms of a harmonical progression be $\dfrac{1}{a+(p-1)d}, \dfrac{1}{a}, \dfrac{1}{a-(p-1)d}:$

we have $$\frac{1}{a-(p-1)d} + \frac{1}{a+(p-1)d} = \frac{2a}{a^2-(p-1)^2d^2},$$

which is manifestly greater than $\dfrac{2}{a}$:

that is, the sum of any two terms of a harmonical series is greater than twice the intermediate mean term: and it is evident that this excess is the greater as they are more remote from it.

Whence we have the sum of n terms of the series greater than n times the middle term: and therefore by the continued increase of n, the sum of any harmonical series may be made greater than any quantity that can be assigned: and hence the sum of the reciprocals of the natural numbers $\dfrac{1}{1} + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \text{\&c.}$ *in infinitum* is indefinitely great.

THE BINOMIAL THEOREM.

36. To find the result arising from multiplying the successive terms of the expansion of $(a+b)^m$, by the corresponding terms of an arithmetical progression whose first term is α and last term λ.

$$\text{Here, } \lambda = \alpha + md, \text{ and } \therefore\ d = \frac{\lambda - \alpha}{m}:$$

$\therefore$ the sum of the terms of the result

$$= \alpha a^m + m\left(\alpha + \frac{\lambda-\alpha}{m}\right) a^{m-1}b + \frac{m(m-1)}{1.2}\left\{\alpha + \frac{2(\lambda-\alpha)}{m}\right\} a^{m-2}b^2 + \&c.$$

$$= \alpha\left\{a^m + ma^{m-1}b + \frac{m(m-1)}{1.2}a^{m-2}b^2 + \&c.\right\}$$

$$+ \frac{\lambda-\alpha}{m}\left\{ma^{m-1}b + 2\frac{m(m-1)}{1.2}a^{m-2}b^2 + \&c.\right\}$$

$$= \alpha(a+b)^m + (\lambda-\alpha)\, b\,(a+b)^{m-1}$$

$$= (a+b)^{m-1}\{a\alpha + b\lambda\}.$$

If b be negative, the result $= (a-b)^{m-1}(a\alpha - b\lambda)$, which becomes $= 0$, whenever $\frac{\alpha}{\lambda} = \frac{b}{a}$.

If $a = b = 1$, and the arithmetical progression be 1, 2, 3, &c., we shall have

$$1 + 2m + 3\frac{m(m-1)}{1.2} + \&c. = (m+2)\,2^{m-1}:$$

$$1 - 2m + 3\frac{m(m-1)}{1.2} - \&c. = 0.$$

37. If a_0, a_1, a_2, &c. a_m, denote the coefficients of an expanded binomial, then will

$$a_0a_r + a_1a_{r+1} + a_2a_{r+2} + \&c. + a_{m-r}a_m = \frac{2m(2m-1)\,\&c.\,(m-r+1)}{1.2.3.\&c.(m+r)}.$$

For, $(1+v)^m = a_0 + a_1v + a_2v^2 + \&c. + a_rv^r + \&c. + a_mv^m$

$$= a_m + a_{m-1}v + a_{m-2}v^2 + \&c. + a_{m-r}v^r + \&c. + a_0v^m;$$

$\therefore\ (1+v)^{2m} =$ the product of these two series

$$\begin{aligned} &= \&c. + a_0 a_r v^{r+m} + \&c. \\ &\quad \&c. + a_1 a_{r+1} v^{r+m} + \&c. \\ &\quad \&c. + a_2 a_{r+2} v^{r+m} + \&c. \end{aligned}$$

but the coefficient of v^{r+m} is that of the $(r+m+1)^{\text{th}}$ term of the expansion of $(1+v)^{2m}$

$$= \frac{2m(2m-1)\ \&c.\ (m-r+1)}{1\,.\,2\,.\,3\,.\,\&c.\ (m+r)}:$$

$$\therefore\ a_0 a_r + a_1 a_{r+1} + a_2 a_{r+2} + \&c. + a_{m-r} a_m = \frac{2m(2m-1)\ \&c.\ (m-r+1)}{1\,.\,2\,.\,3\,.\,\&c.\ (m+r)}.$$

If $r=0$, we have $a_r = a_0$, $a_{r+1} = a_1$, &c.,

whence, $a_0^2 + a_1^2 + a_2^2 + \&c. + a_m^2 = \dfrac{1\,.\,2\,.\,3\,.\,\&c.\ 2m}{(1\,.\,2\,.\,3\,.\,\&c.\ m)^2}$.

If $r=1$, we have $a_r = a_1$, $a_{r+1} = a_2$, &c.,

$$\therefore\ a_0 a_1 + a_1 a_2 + a_2 a_3 + \&c. + a_{m-1} a_m = \frac{1\,.\,2\,.\,3\,.\,\&c.\ (2m-1)}{(m+1)\{1\,.\,2\,.\,3\,.\,\&c.\,(m-1)\}^2}:$$

and similarly of others.

38. Using the same notation, we have

$$(a+b\sqrt{-1})^m + (a-b\sqrt{-1})^m = 2\{a_1 - a_3 + a_5 - \&c.\},$$

$$(a+b\sqrt{-1})^m - (a-b\sqrt{-1})^m = 2\sqrt{-1}\{a_2 - a_4 + a_6 - \&c.\}:$$

and thus the values of $a_1 - a_3 + a_5 - \&c.$, and $a_2 - a_4 + a_6 - \&c.$ are expressed in symbolical forms.

THE MULTINOMIAL THEOREM.

39. Def. The *Multinomial Theorem* is a formula for expanding any power of an algebraical quantity consisting of more than two terms.

40. *In the expansion of* $(\mathrm{a}+\mathrm{b}+\mathrm{c}+\&c.)^{\mathrm{m}}$, *to find the coefficient of the literal product* $\mathrm{a}^{\alpha}\mathrm{b}^{\beta}\mathrm{c}^{\gamma}$ &c.

For $b+c+d+$ &c. put b', and let $m=a+\beta'$: then, one term of the expansion of $(a+b')^m$ will be

$$\frac{m(m-1)(m-2)\,\&\text{c.}\,(m-\beta'+1)}{1.2.3.\,\&\text{c.}\,\beta'}\,a^{m-\beta'}b'^{\beta'}$$

$$=\frac{1.2.3.\,\&\text{c.}\,m}{(1.2.3.\,\&\text{c.}\,a)(1.2.3.\,\&\text{c.}\,\beta')}\,a^{a}b'^{\beta'}.$$

Again, for b' put $b+c'$, and let $\beta'=\beta+\gamma'$: then in the expansion of $(b+c')^{\beta'}$, one term will be

$$\frac{1.2.3.\,\&\text{c.}\,\beta'}{(1.2.3.\,\&\text{c.}\,\beta)(1.2.3.\,\&\text{c}\,\gamma')}\,b^{\beta}c'^{\gamma'},$$

wherein $a+\beta+\gamma'=m$: and by combining this with the former, we have one of the terms of the expansion of

$$(a+b+c')^m$$

$$=\frac{1.2.3.\,\&\text{c.}\,m}{(1.2.3.\,\&\text{c.}\,a)(1.2.3.\,\&\text{c.}\,\beta)(1.2.3.\,\&\text{c.}\,\gamma')}\,a^{a}b^{\beta}c'^{\gamma'}:$$

and proceeding as above, and putting $c+d'$ for c', and $\gamma+\delta'$ for γ', we shall obtain the general term of $(a+b+c+\&\text{c.})^m$

$$=\frac{1.2.3.\,\&\text{c.}\,m}{(1.2.3.\,\&\text{c.}\,a)(1.2.3.\,\&\text{c.}\,\beta)(1.2.3.\,\&\text{c.}\,\gamma)\,\&\text{c.}}\,a^{a}\,b^{\beta}\,c^{\gamma}\,\&\text{c.}:$$

where the quantities a, β, γ, &c. are subject to the condition $a+\beta+\gamma+\&\text{c.}=m$.

41. *To find the terms of the expansion of*

$$(a+bx+cx^2+\&\text{c.}+bx^p)^m.$$

The general term of $(a+b+c+\&\text{c.})^m$ being already found, if we put bx, cx^2, &c., in the places of b, c, &c., we shall have the general term of the expansion of

$$(a+bx+cx^2+\&\text{c.}+bx^p)^m$$

$$=\frac{1.2.3.\,\&\text{c.}\,m}{(1.2.3.\&\text{c.}a)(1.2.3.\&\text{c.}\beta)(1.2.3.\&\text{c.}\gamma)\&\text{c.}}\,a^{a}b^{\beta}c^{\gamma}\,\&\text{c.}\,x^{\beta+2\gamma+\&\text{c.}}:$$

and all the terms in which the index of x is $\beta+2\gamma+$ &c. may

be derived from this, by giving to a, β, γ, &c. all the different positive integral values of which they are capable, consistently with the limitation, $a+\beta+\gamma+\&c.=m$: and if $\beta+2\gamma+\&c.$ be assumed $=r$, the two equations of condition will be

$$a+\beta+\gamma+\&c.=m:$$
$$\beta+2\gamma+3\delta+\&c.=r.$$

If $\beta+\gamma+\delta+\&c.=\phi$, the entire coefficient of x^r will be obtained by forming all the possible literal products and corresponding coefficients, subject to the conditions expressed by

$$\beta+\gamma+\delta+\&c.=\phi,\ \text{and}\ \beta+2\gamma+3\delta+\&c.=r.$$

Ex. Find the term of the expansion of $(a-bx+cx^2)^{12}$, which involves x^8.

Here, $\beta+2\gamma=8$, and $a+\beta+\gamma=12$:

whence, $a=4+\gamma$, and $\beta=8-2\gamma$:

$$\begin{aligned} \therefore\ \text{if}\ \gamma&=0,\ \text{we have}\ a=4,\ \text{and}\ \beta=8: \\ \gamma&=1,\ \ldots\ldots\ldots\ a=5,\ \ldots.\ \beta=6: \\ \gamma&=2,\ \ldots\ldots\ldots\ a=6,\ \ldots.\ \beta=4: \\ \gamma&=3,\ \ldots\ldots\ldots\ a=7,\ \ldots.\ \beta=2: \\ \gamma&=4,\ \ldots\ldots\ldots\ a=8,\ \ldots.\ \beta=0: \end{aligned}$$

whence, by substitution in the general formula, the term required will be found to be

$$\left\{\frac{12.11.10.9}{1.2.3.4}a^4b^8+\frac{12.11.10.9.8.7}{1.2.3.4.5}a^5b^6c\right.$$
$$+\frac{12.11.10.9.8.7}{(1.2.3.4)(1.2)}a^6b^4c^2+\frac{12.11.10.9.8}{(1.2)(1.2.3)}a^7b^2c^3$$
$$\left.+\frac{12.11.10.9}{1.2.3.4}a^8c^4\right\}x^8.$$

42. If $a_0+a_1x+a_2x^2+a_3x^3+\&c.+a_px^p$ be the proposed multinomial, the theorem for its expansion may be exhibited in a different form.

$$\text{Let } (a_0 + a_1 x + a_2 x^2 + \&c.)^m = t_0 + t_1 x + t_2 x^2 + t_3 x^3 + \&c.:$$

$$\therefore\ (a_0 + a_1 y + a_2 y^2 + \&c.)^m = t_0 + t_1 y + t_2 y^2 + t_3 y^3 + \&c.:$$

$$\therefore\ \frac{u^m - v^m}{u - v} = \frac{t_1(x-y) + t_2(x^2-y^2) + t_3(x^3-y^3) + \&c.}{a_1(x-y) + a_2(x^2-y^2) + a_3(x^3-y^3) + \&c.}$$

$$= \frac{t_1 + t_2(x+y) + t_3(x^2+xy+y^2) + \&c.}{a_1 + a_2(x+y) + a_3(x^2+xy+y^2) + \&c.}:$$

therefore, by making $y = x$, and consequently $v = u$, we have

$$\frac{t_1 + 2t_2 x + 3t_3 x^2 + \&c.}{a_1 + 2a_2 x + 3a_3 x^2 + \&c.} = mu^{m-1}, \text{ by article (35),}$$

$$= \frac{mu^m}{u} = \frac{m(t_0 + t_1 x + t_2 x^2 + t_3 x^3 + \&c.)}{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \&c.}:$$

whence, multiplying out, we obtain

$$\left.\begin{array}{r} a_0 t_1 + 2a_0 t_2 \\ +\ a_1 t_1 \end{array}\right\} x \quad \left.\begin{array}{r} +\ 3a_0 t_3 \\ +\ 2a_1 t_2 \\ +\ a_2 t_1 \end{array}\right\} x^2 \quad \left.\begin{array}{r} +\ 4a_0 t_4 \\ +\ 3a_1 t_3 \\ +\ 2a_2 t_2 \\ +\ a_3 t_1 \end{array}\right\} x^3 + \&c.$$

$$= ma_1 t_0 + \left.\begin{array}{r} ma_1 t_1 \\ +\ 2ma_2 t_0 \end{array}\right\} x \quad \left.\begin{array}{r} +\ ma_1 t_2 \\ +\ 2ma_2 t_1 \\ +\ 3ma_3 t_0 \end{array}\right\} x^2 \quad \left.\begin{array}{r} +\ ma_1 t_3 \\ +\ 2ma_2 t_2 \\ +\ 3ma_3 t_1 \\ +\ 4ma_4 t_0 \end{array}\right\} x^3 + \&c.:$$

and therefore by equating coefficients, we find

$$t_1 = m\frac{a_1}{a_0} t_0:$$

$$t_2 = 2m\frac{a_2}{2a_0} t_0 + (m-1)\frac{a_1}{2a_0} t_1:$$

$$t_3 = 3m\frac{a_3}{3a_0} t_0 + (2m-1)\frac{a_2}{3a_0} t_1 + (m-2)\frac{a_1}{3a_0} t_2: \ \&c.:$$

whence, $\{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \&c. + a_p x^p\}^m$

$$= a_0^m + m a_1 t_0 \left| \frac{x}{a_0} + (2m - 0) a_2 t_0 \atop + (1m - 1) a_1 t_1 \right| \frac{x^2}{2a_0} + \begin{matrix} (3m - 0) a_3 t_0 \\ + (2m - 1) a_2 t_1 \\ + (1m - 2) a_1 t_2 \end{matrix} \left| \frac{x^3}{3a_0} \right.$$

$$+ \left.\begin{matrix} (4m - 0) a_4 t_0 \\ + (3m - 1) a_3 t_1 \\ + (2m - 2) a_2 t_2 \\ + (1m - 3) a_1 t_3 \end{matrix} \right| \frac{x^4}{4a_0} + \&c.:$$

where the law of derivation is manifest, t_0, t_1, t_2, t_3, &c., being the terms of the expansion in which the indices of x are 0, 1, 2, 3, &c., respectively.

If $a_2 = a_3 = \&c. = 0$, we have the expansion of a binomial obtained by a similar process.

THE EXPONENTIAL THEOREM.

43. Assume $a^x = 1 + Ax + Bx^2 + Cx^3 + \&c.$, where the first term $= 1$, since $a^0 = 1$:

$$\therefore\ a^{x+h} = 1 + A(x+h) + B(x+h)^2 + C(x+h)^3 + \&c.$$

$$= 1 + Ax + Bx^2 + Cx^3 + \&c.$$

$$+ \{A + 2Bx + 3Cx^2 + 4Dx^3 + \&c.\}\, h + \&c.:$$

$$\text{also, } a^{x+h} = a^x \{1 + Ah + Bh^2 + Ch^3 + \&c.\}:$$

whence, equating the coefficients of h in these expressions,

$$\text{we have } A + 2Bx + 3Cx^2 + 4Dx^3 + \&c. = Aa^x$$

$$= A\{1 + Ax + Bx^2 + Cx^3 + \&c.\}:$$

$$\therefore\ A = A,\ 2B = A^2, \text{ and } B = \frac{A^2}{1.2}:$$

$$3C = AB = \frac{A^3}{1.2}, \text{ and } C = \frac{A^3}{1.2.3}:$$

$$4D = AC = \frac{A^4}{1.2.3}, \text{ and } D = \frac{A^4}{1.2.3.4}: \&c.:$$

$$\text{whence, } a^x = 1 + \frac{A}{1}x + \frac{A^2}{1.2}x^2 + \frac{A^3}{1.2.3}x^3 + \&c.:$$

and the value of A is proved to be equivalent to

$$(a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \&c. \textit{ in infinitum,}$$

by finding the coefficient of x in the expansion of

$$a^x = \{1 + (a-1)\}^x.$$

See the Author's *Trigonometry*, for the application of this expansion to the calculation of Logarithms.

THE LOGARITHMIC SERIES.

44. If we suppose $y = 1 + x$: then $\log y = \log(1+x)$: and since when $x = 0$, $\log 1 = 0$, we may assume

$$\log(1+x) = A_1x + A_2x^2 + A_3x^3 + \&c.:$$

$$\therefore \log(1+x+h) = \log(1+x)\left(1 + \frac{h}{1+x}\right)$$

$$= \log(1+x) + \log\left(1 + \frac{h}{1+x}\right)$$

$$= A_1(x+h) + A_2(x+h)^2 + A_3(x+h)^3 + \&c.:$$

$$\text{but } \log(1+x) + \log\left(1 + \frac{h}{1+x}\right)$$

$$= A_1x + A_2x^2 + A_3x^3 + \&c.$$

$$+ A_1\frac{h}{1+x} + A_2\left(\frac{h}{1+x}\right)^2 + A_3\left(\frac{h}{1+x}\right)^3 + \&c.:$$

$$\text{whence, } A_1 + 2A_2x + 3A_3x^2 + \&c. = \frac{A_1}{1+x}$$

$$= A_1 - A_1x + A_1x^2 - A_1x^3 + \&c.:$$

$$\therefore 2A_2 = -A_1, \quad 3A_3 = A_1, \quad 4A_4 = -A_1, \quad \&c.:$$

$$\therefore \; A_2 = -\tfrac{1}{2}A_1, \quad A_3 = \tfrac{1}{3}A_1, \quad A_4 = \tfrac{1}{4}A_1, \text{ \&c.,}$$

$$\text{and } \log(1+x) = A_1\{x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \text{\&c.}\}:$$

also, if a be the base of the system, we have

$$1 = \log a = A_1\{(a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \text{\&c.}\}$$

by this formula: whence, A_1 is determined:

$$\text{and } \therefore \log(1+x) = \frac{x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \text{\&c.}}{(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \text{\&c.}}.$$

If the denominator be assumed $= 1$, this formula will be much simplified, and the value of a to answer this purpose may easily be found.

For, if e be this value, we have from the preceding article

$$e^x = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \text{\&c.}:$$

$$\therefore \; e = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \text{\&c.,}$$

whose sum to 10 terms will be found to be 2.71828 &c.; and denoting the logarithm to the base e by $\log_e$, we have

$$\log_e(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \text{\&c.}:$$

which is easily remembered, and is of great practical utility in the higher departments of Mathematical Science. See the *Trigonometry*, for various modifications of this series.

VARIATIONS AND COMBINATIONS.

45. We have seen in article (223) that of m things,

$$C_r = \frac{m(m-1) \text{ \&c. } (m-r+1)}{1.2.3.\text{\&c. } r}:$$

whence assigning to r, the values 1, 2, 3, &c., m, we have

$$C_1 + C_2 + C_3 + \text{\&c.} + C_m = m + \frac{m(m-1)}{1.2} + \frac{m(m-1)(m-2)}{1.2.3} + \text{\&c.}$$

$$= (1+1)^m - 1 = 2^m - 1:$$

and if the m *single* things be omitted, the sum of all the combinations $= 2^m - (m + 1)$.

Thus, if $m = 5$, the number of all the combinations formed by taking 2, 3, 4, 5 at a time $= 2^5 - 6 = 26$.

Also, since $(C_1 + C_3 + C_5 + \&c.) - (C_2 + C_4 + C_6 + \&c.) = 1$, it follows that the total number of odd combinations thus formed, is greater by 1 than the total number of even combinations.

46. If there be two sets of things consisting of m and m' individuals, to find the number of combinations, that can be formed by selecting out of them r and r' respectively.

$$\text{Here, } C_r = \frac{m(m-1)\ \&c.\ (m-r+1)}{1\,.\,2\,.\,3\,.\ \&c.\ r}:$$

$$C_{r'} = \frac{m'(m'-1)\ \&c.\ (m'-r'+1)}{1\,.\,2\,.\,3\,.\ \&c.\ r'}:$$

and since each of the former may be combined with each of the latter, the required number will manifestly

$$= \frac{m(m-1)\ \&c. - (m-r+1)}{1\,.\,2\,.\,3\,.\ \&c.\ r} \times \frac{m'(m'-1)\ \&c.\ (m'-r'+1)}{1\,.\,2\,.\,3\,.\ \&c.\ r'},$$

where each combination consists of $r + r'$ individuals.

Similarly, whatever be the number of sets of things, the number of combinations formed in the same way will be

$$= C_r \times C_{r'} \times C_{r''} \times \&c.$$

If $r = r' = r'' = \&c. = 1$, or one thing be taken out of each set, the number of combinations will be

$$m \times m' \times m'' \times \&c.$$

Ex. There are four sets of things, consisting of three, four, five, and six individuals respectively: what number of different collections can there be made by always taking one out of each set?

Here, $m = 3$, $m' = 4$, $m'' = 5$ and $m''' = 6$:

$\therefore$ the required number $= 3 . 4 . 5 . 6 = 360$.

If there be the same number of things in each set, $m = m' = m'' = m''' =$ &c.; and if one be taken out of each, the number of combinations will be m^p, if p be the number of sets.

If p be made equal to 1, 2, 3, &c., in succession, we shall have

$$m + m^2 + m^3 + \&c. + m^p = m\left(\frac{m^p - 1}{m - 1}\right).$$

This is evidently the number of variations of m things, taken p together, where each thing may be found 1, 2, 3, &c., p times in each variation.

Ex. How many combinations of three numbers can be obtained by throwing three dice?

Each die having six faces, the question is to determine how many combinations of three things can be formed out of three sets, each consisting of six individuals, by always selecting one out of each set:

$\therefore$ $m = 6$, $m' = 6$, $m'' = 6$, and the required number $= 216$.

47. COR. If $m = 2$ and $r = 1$, we have the number of combinations $= 2 \times \dfrac{m'(m' - 1) \text{ \&c. } (m' - r' + 1)}{1 . 2 . 3 . \text{\&c. } r'}$, which is manifestly the number of combinations formed by taking each one of the first set of things singly, and combining them with all the combinations of the second set taken r' together.

If $m = 3$ and $r = 2$, we shall have the number of combinations $= \dfrac{3 . 2}{1 . 2} \times \dfrac{m'(m' - 1) \text{ \&c. } (m' - r' + 1)}{1 . 2 . 3 . \text{\&c. } r'}$: and this is the number of combinations which can be formed by selecting two out of the first set, and combining them in all possible ways with those of the second set.

Similarly, of other values of m and r.

48. To find the number of variations corresponding to the number of combinations in the preceding articles.

If $r + r' = p$, it is evident that every combination in $C_r \times C_{r'}$, admits of $1 . 2 . 3 .$ &c. p variations: whence the number of variations required will be $1 . 2 . 3 .$ &c. $p \times C_r \times C_{r'}$.

49. To find the number of combinations of m things, taken 1 and 1, 2 and 2, &c., m and m together, when there are p of one sort, q of another, r of another, &c.

The question is evidently to find the number of divisors of an expression of the form $a^p b^q c^r$ &c.: which, by article (381), is $(p + 1)(q + 1)(r + 1)$ &c.: and subtracting 1 which does not involve any of the things, from it, we have the required number $= (p + 1)(q + 1)(r + 1)$ &c. $- 1$.

Also, the number of combinations containing any particular thing as b, will evidently be the same as the number of terms in the product

$$(1 + a + a^2 + \&c. + a^p)(b + b^2 + b^3 + \&c. + b^q)(1 + c + c^2 + \&c. + c^r)\ \&c.$$

$$= (p + 1)\, q\, (r + 1)\ \&c.$$

See *Francœur's* Complete Course of Pure Mathematics.

50. To find the number of homogeneous products of the r^{th} order, that can be formed out of the m simple quantities a, b, c, &c.

Let there be m infinite series multiplied together, so that

$$(1 + ax + a^2x^2 + \&c.)(1 + bx + b^2x^2 + \&c.)(1 + cx + c^2x^2 + \&c.)\ \&c.$$

$$= 1 + A_1 x + A_2 x^2 + \&c. + A_r x^r + \&c.:$$

then $A_r =$ the sum of the products of r dimensions that can be formed out of the m quantities a, b, c, &c.: whence, if $a = b = c =$ &c., we shall have $A_r = a^r \times$ the *number* of homogeneous products of the r^{th} order:

$$\text{but } 1 + ax + a^2x^2 + \&c. \textit{ in infinitum} = \frac{1}{1 - ax}:$$

$$\therefore\ 1 + A_1 x + A_2 x^2 + \&c. + A_r x^r + \&c. = (1 - ax)^{-m}$$

$$= 1 + max + \&c. + \frac{m(m+1)\ \&c.\ (m+r-1)}{1\,.\,2\,.\,3\,.\ \&c.\ r}\, a^r x^r + \&c.:$$

whence the number of homogeneous products of the r^{th} order, formed out of m things, will be

$$= \frac{m(m+1)\ \&c.\ (m+r-1)}{1\,.\,2\,.\,3\,.\ \&c.\ r}.$$

51. COR. From the last article, by making m equal to 2, 3, 4, &c., in succession, we learn that the number of terms in the developement of $(a+b)^r$ is $r+1$: the number in the developement of $(a+b+c)^r$ is $\dfrac{(r+1)(r+2)}{1\,.\,2}$: the number in the developement of $(a+b+c+d)^r$ is $\dfrac{(r+1)(r+2)(r+3)}{1\,.\,2\,.\,3}$: and so on.

SCALES OF NOTATION.

52. Given the number of digits contained in each of two numbers, to find the number of digits in their product.

Let P and Q consist of p and q digits respectively:

$$\text{then, } P = a_{p-1} r^{p-1} + a_{p-2} r^{p-2} + \&c. + a_1 r + a_0,$$

$$\text{and } Q = b_{q-1} r^{q-1} + b_{q-2} r^{q-2} + \&c. + b_1 r + b_0:$$

whence, multiplying these quantities together, we obtain

$$PQ = a_{p-1} b_{q-1} r^{p+q-2} + a_{p-2} b_{q-1} r^{p+q-3} + \&c.$$
$$+ a_{p-1} b_{q-2} r^{p+q-3} + \&c.:$$

from which we infer by article (348), that PQ must always consist of $p+q-1$ digits at least: also, since each of the digits is necessarily less than r, the product of any two of them must be less than r^2, but may be greater than r, and thence it follows that the highest power of r in PQ must be less than r^{p+q}, but may be equal to r^{p+q-1}: that is, the product PQ cannot comprise more digits than $p+q$.

53. Cor. If the three numbers P, Q, S consist of p, q, s digits respectively: then, since PQ contains $p+q-1$ or $p+q$ digits, it is obvious that PQS will comprise either

$$(p+q-1)+s-1,\ (p+q-1)+s, \text{ or } p+q+s \text{ digits}:$$

that is, PQS may consist of $p+q+s-2$, $p+q+s-1$, or $p+q+s$ digits: and the same kind of reasoning may be extended to the product of any number of quantities whatever.

54. Given the number of digits comprised in each of two numbers, to find the number of digits in their quotient.

Let P and Q denote the two numbers consisting of p and q digits respectively, whereof P is the greater: and let M be the quotient arising from the division, so that

$$\frac{P}{Q}=M, \text{ or } P=QM:$$

then, since P comprises p digits, it follows that QM must contain the same number: let M contain x digits: then the number of digits in QM cannot be

$$>q+x, \text{ nor } <q+x-1:$$

whence x cannot be $<p-q$, nor $>p-q+1$, which defines the number of digits in $\frac{P}{Q}$.

55. Given the number of digits constituting any number, to find the number of digits in its square, cube, &c.

Let P consist of p digits: then the number of digits in P^2 or $P\times P$, cannot be

$$<p+p-1, \text{ or } 2p-1, \text{ nor } >p+p \text{ or } 2p:$$

again, the number of digits in P^3, or $P\times P\times P$, may be

$$p+p+p-2, \text{ or } p+p+p-1, \text{ or } p+p+p:$$

that is, P^3 may comprise $3p-2$, $3p-1$, or $3p$ digits: and a continuation of the same reasoning will prove that the number of digits in P^m may be

$$mp-(m-1),\ mp-(m-2),\ mp-(m-3), \text{ \&c.}, mp.$$

56. Given the number of digits forming any number, to find the number of digits in its square root, cube root, &c.

Let P consist of p digits, and suppose $\sqrt{P}$ to comprise x digits: then, by the last article, the number of digits in P cannot be

$$< 2x - 1, \text{ nor } > 2x:$$

$$\text{that is, } p \text{ is not } < 2x - 1, \text{ nor } > 2x:$$

$$\therefore\ x \text{ is not } > \frac{p+1}{2}, \text{ nor } < \frac{p}{2}.$$

Again, if $\sqrt[3]{P}$ comprise y digits, P may have $3y - 2$, $3y - 1$, or $3y$ digits:

that is, we must have $p = 3y - 2$, or $p = 3y - 1$, or $p = 3y$:

$$\text{and therefore } y = \frac{p+2}{3}, \text{ or } y = \frac{p+1}{3}, \text{ or } y = \frac{p}{3}:$$

and so on, for higher roots.

57. Cor. From this proposition are immediately deduced the rules for *pointing* in the extraction of the square, cube, &c. roots of numbers, as laid down in the second Chapter.

58. Given a number expressed in any scale of notation, to find the remainder arising from its division by any number either greater or less than the base of the system.

Let $N = a_m r^m + a_{m-1} r^{m-1} + \&\text{c.} + a_2 r^2 + a_1 r + a_0$:

thence, since $r = r - d + d$, we shall have

$$N = a_m \{(r-d) + d\}^m + a_{m-1} \{(r-d) + d\}^{m-1} + \&\text{c.}$$
$$+ a_2 \{(r-d) + d\}^2 + a_1 \{(r-d) + d\} + a_0:$$

from which, if the expansions be effected, we have

$$N = a_0 + a_1 d + a_2 d^2 + \&\text{c.} + a_{m-1} d^{m-1} + a_m d^m + P,$$

where P involves $r - d$, and its powers combined with the indices m, $m - 1$, $m - 2$, &c., and the powers of d:

whence, if N be divided by $r-d$, it will leave the same remainder as will be obtained by dividing by $r-d$, the quantity

$$a_0 + a_1 d + a_2 d^2 + \&c. + a_{m-1} d^{m-1} + a_m d^m.$$

If $d=1$, we shall have

$$N = a_0 + a_1 + a_2 + \&c. + a_{m-1} + a_m + P:$$

from which it appears that when N is divided by $r-1$, it leaves the same remainder as when the sum of its digits is divided by $r-1$, as before proved.

If $d=-1$, we have immediately,

$$N = a_0 - a_1 + a_2 - \&c. \mp a_{m-1} \pm a_m + P:$$

and from this it follows that both members of the equality when divided by $r+1$, leave the same remainder.

If $d=2$, the general formula gives

$$N = a_0 + a_1 2 + a_2 2^2 + \&c. + a_{m-1} 2^{m-1} + a_m 2^m + P,$$

from which we draw the same conclusion as before: and if $r=10$, we shall have N divisible by 8, wherever $a_0 + 2a_1 + 4a_2$ is so divisible, since the succeeding terms are all multiples of 2^3 or 8.

If $d=-2$, we have similarly

$$N = a_0 - a_1 2 + a_2 2^2 - \&c. \mp a_{m-1} 2^{m-1} \pm a_m 2^m + P,$$

from which we conclude that in the common scale, if the digits beginning with the place of units, be successively multiplied by 1, 2, 2^2, 2^3, &c., and the difference of the odd and even terms thus arising be divisible by 12, the number itself is divisible by 12.

By assigning to d, the values ± 3, ± 4, &c., criteria of divisibility by 7, 13 : 6, 14, &c. will be immediately deduced.

PRIME NUMBERS.

59. *M. Fermat's Theorem.* If m be a prime number, and N be not divisible by m, then will $N^{m-1}-1$ be divisible by m.

For, by article (395), we have seen that

$(1+x)^m-(1+x^m)$ is divisible by m, whenever x is integral:

therefore, assuming $1+x=N$, we shall have

$N^m-1-(N-1)^m$ divisible by m, which suppose $=mQ_1$,

so that $N^m-N=(N-1)^m-(N-1)+mQ_1$;

similarly, $(N-1)^m-(N-1)=(N-2)^m-(N-2)+mQ_2$;

$(N-2)^m-(N-2)=(N-3)^m-(N-3)+mQ_3$;

&c.

and continuing this process, we obviously at length arrive at

$$1^m-1=(N-N)^m-(N-N)+mQ_n, \text{ or } 0=mQ_n;$$

whence, by addition, $N^m-N=m\{Q_1+Q_2+Q_3+\text{\&c.}+Q_{n-1}\}$, which is therefore divisible by m:

but N^m-N being $=N(N^{m-1}-1)$, whereof N is prime to m, it remains only that $N^{m-1}-1$ is divisible by m.

60. Cor. 1. Since N^m-N is divisible by m, it manifestly follows that N^m, when divided by m, leaves the same remainder as N divided by m leaves.

61. Cor. 2. Because all the numbers 1, 2, 3, 4, &c., $m-1$, are prime to m, each of them when substituted for N will render the expression $\dfrac{N^{m-1}-1}{m}=n$, a whole number: and since $m-1$ is necessarily even, there will evidently be $m-1$ values of N comprised between the magnitudes $-\frac{1}{2}m$ and $\frac{1}{2}m$, which answer the same condition; or, N may be any one of numbers

$$\pm 1,\ \pm 2,\ \pm 3,\ \pm 4,\ \text{\&c.},\ \pm\tfrac{1}{2}(m-1).$$

62. Cor. 3. Having proved that $\dfrac{N^{m-1}-1}{m}$ is a whole number, as n, we shall obviously have N^{m-1} of the form $mn+1$: and consequently every power of N whose exponent increased by 1 is a prime number, will be of the form mn or $mn+1$, according as N is or is not divisible by m.

Thus, N^2 is of the form $3n$ or $3n+1$:

N^4 .... $5n$ or $5n+1$:

N^6 .... $7n$ or $7n+1$:

N^{10} .... $11n$ or $11n+1$:

&c.

63. Cor. 4. Since $m-1$ is an even number, we shall have

$$N^{m-1}-1=\left\{N^{\frac{1}{2}(m-1)}+1\right\}\left\{N^{\frac{1}{2}(m-1)}-1\right\},$$

one of the latter factors of which must manifestly be divisible by m, and consequently

$$N^{\frac{1}{2}(m-1)} \text{ is of the form } mn \pm 1;$$

that is, every power of N, the double of whose exponent increased by 1 becomes a prime number, is of the form mn or $mn \pm 1$, according as N is divisible by m or not.

Thus, N^2 is of the form $5n$ or $5n \pm 1$:

N^3 .... $7n$ or $7n \pm 1$:

N^5 .... $11n$ or $11n \pm 1$:

N^6 .... $13n$ or $13n \pm 1$:

&c.

64. *Sir John Wilson's Theorem.* If m be a prime number, then will the continued product $1 . 2 . 3 . \&c. (m-1)$, when augmented by 1, be divisible by m.

By means of the *Differential Calculus*, article (344), or by the *Calculus of Finite Differences*, it is easily demonstrated that

$$1 . 2 . 3 . \&c.\ n = n^n - \frac{n}{1}(n-1)^n + \frac{n(n-1)}{1 . 2}(n-2)^n - \&c.,$$

to n terms, whatever whole number n may be:

whence, if n be assumed $= m - 1$, we shall have

$$1 . 2 . 3 . \&c.\ (m-1) = (m-1)^{m-1} - \frac{(m-1)}{1}(m-2)^{m-1}$$

$$+ \frac{(m-1)(m-2)}{1 . 2}(m-3)^{m-1} - \&c. \text{ to } m-1 \text{ terms:}$$

and since m is prime to each of the quantities $m-1$, $m-2$, $m-3$, &c., we obtain from *Fermat's* theorem, the following results:

$$(m-1)^{m-1} = mQ_1 + 1;$$

$$(m-2)^{m-1} = mQ_2 + 1;$$

$$(m-3)^{m-1} = mQ_3 + 1;$$

&c.

$$\therefore\ 1 . 2 . 3 . \&c.\ (m-1) = 1 - \frac{m-1}{1} + \frac{(m-1)(m-2)}{1 . 2} - \&c.$$

$$\text{to } m-1 \text{ terms} + m\left\{Q_1 - \frac{m-1}{1}Q_2 + \frac{(m-1)(m-2)}{1 . 2}Q_3 - \&c.\right\}:$$

but, since $m-1$ is necessarily an even number, we shall have

$$1 - \frac{m-1}{1} + \frac{(m-1)(m-2)}{1 . 2} - \&c., \text{ to } m-1 \text{ terms}$$

$$= (1-1)^{m-1} - 1 = -1:$$

$$\therefore\ 1.2.3.\&c.(m-1) = m\left\{Q_1 - \frac{m-1}{1}Q_2 + \frac{(m-1)(m-2)}{1 . 2}Q_3 - \&c.\right\} - 1;$$

whence,

$$1.2.3.\&c.(m-1) + 1 = m\left\{Q_1 - \frac{m-1}{1}Q_2 + \frac{(m-1)(m-2)}{1 . 2}Q_3 - \&c.\right\},$$

which is obviously divisible by m.

65. COR. 1. Since $m-1$ is an even number, the continued product $1.2.3.\&c.(m-1)$ is equivalent to

$$1(m-1)2(m-2)3(m-3).\&c.\left\{\tfrac{1}{2}(m-1)\right\}^2:$$

which, when divided by m, manifestly leaves the same remainder as

$$\pm\left\{1.2.3.\&c.\tfrac{1}{2}(m-1)\right\}^2,$$

wherein the upper or lower sign is applicable, according as $\frac{1}{2}(m-1)$ is even or odd, or according as m is of the form $4n+1$ or $4n-1$: consequently, in the former case we shall have

$$\left\{1.2.3.\&c.\tfrac{1}{2}(m-1)\right\}^2+1,\text{ divisible by } m:$$

and in the latter,

$$\left\{1.2.3.\&c.\tfrac{1}{2}(m-1)\right\}^2-1,\text{ divisible by } m.$$

66. COR. 2. By means of the former of these results, it appears that every prime number of the form $4n+1$ will divide the sum of two squares without a remainder.

67. COR. 3. Since, in the latter of the expressions deduced above, we have

$$\left\{1.2.3.\&c.\tfrac{1}{2}(m-1)\right\}^2-1$$

$$=\left\{1.2.3.\&c.\tfrac{1}{2}(m-1)+1\right\}\left\{1.2.3.\&c.\tfrac{1}{2}(m-1)-1\right\},$$

it manifestly follows, that when m is a prime number of the form $4n-1$, it will divide one or other of these factors.

68. COR. 4. Because $\dfrac{1.2.3.\&c.(m-1)+1}{m}$, is an integral quantity, whenever m is a prime number, we shall also have

$$\frac{1.2.(m-2).3.\&c.(m-3)(m-1)+1}{m},$$

and $\therefore$ $\dfrac{1.2^2.3.\&c.(m-3)(m-1)-1}{m}$, an integral quantity:

similarly, $\dfrac{1 . 2^2 . 3^2 . \&c. (m-4)(m-1)+1}{m}$, &c may be proved to be integral.

Sir J. Wilson's theorem furnishes a criterion for deciding whether a proposed number be prime or not, but the magnitude to which the continued product soon rises, renders it of much less practical utility than the one given in (393).

69. *If* $\mathrm{N} = \mathrm{a^p b^q c^r}$ &c., *then will the number of integers less than* N, *and prime to it, be expressed by*

$$(a-1)(b-1)(c-1) \text{ \&c. } a^{p-1} b^{q-1} c^{r-1} \text{ \&c.}:$$

$$\text{or } N\left(\frac{a-1}{a}\right)\left(\frac{b-1}{b}\right)\left(\frac{c-1}{c}\right) \text{ \&c.}$$

First, if we suppose $q = r = \&c. = 0$, and $\therefore N = a^p$, we shall manifestly have the a^p whole numbers 1, 2, 3, 4, &c., a^p, not greater than N: and in this series it is obvious that every a^{th} term is a multiple of a, and therefore not prime to N, the number of such terms as a, $2a$, $3a$, $4a$, &c., $a^{p-1}a$, evidently being a^{p-1}:

whence, of these the number which are not multiples of a is

$$= a^p - a^{p-1} = (a-1)\, a^{p-1} = N\left(\frac{a-1}{a}\right).$$

Next, let $r = \&c. = 0$, or $N = a^p b^q$; then it is evident that the a^{p-1} multiples of a comprised in the numbers 1, 2, 3, 4, &c., a^p, being combined with each of the terms 1, b, b^2, &c., b^q, will, by (381), give $a^{p-1} b^q$ numbers less than N divisible by a: similarly, we shall have $a^p b^{q-1}$ numbers less than N divisible by b, and $a^{p-1} b^{q-1}$ numbers divisible by ab; and it is manifest that these latter $a^{p-1} b^{q-1}$ numbers are likewise included in the two former sets: whence it follows that there are less than N

$a^{p-1} b^q - a^{p-1} b^{q-1} = (b-1)\, a^{p-1} b^{q-1}$ numbers divisible by a only, and

$a^p b^{q-1} - a^{p-1} b^{q-1} = (a-1)\, a^{p-1} b^{q-1}$ numbers divisible by b

therefore the number of numbers less than N which involve neither a, b nor ab, will evidently be

$$= a^p b^q - (b-1)\,a^{p-1}b^{q-1} - (a-1)\,a^{p-1}b^{q-1} - a^{p-1}b^{q-1}$$

$$= (ab - a - b + 1)\,a^{p-1}b^{q-1}$$

$$= (a-1)(b-1)\,a^{p-1}b^{q-1} = N\left(\frac{a-1}{a}\right)\left(\frac{b-1}{b}\right):$$

and the same mode of reasoning, when $N = a^p b^q c^r$ &c., will lead to the conclusion, that the number of integers, unity included, less than N and prime to it, is expressed by

$$(a-1)(b-1)(c-1)\text{ \&c. } a^{p-1}b^{q-1}c^{r-1}\text{ \&c.}:$$

$$\text{or } N\left(\frac{a-1}{a}\right)\left(\frac{b-1}{b}\right)\left(\frac{c-1}{c}\right)\text{ \&c.}$$

70. Cor. Hence, in (394), the number of forms of prime numbers to any given modulus may be determined.

For, let $4A$ be the modulus used: then it is manifest that the number of forms will be equal to the number of integers that are less than $2A$ and prime to it:

now, if $2A = a^p b^q c^r$ &c., we have just seen that the number of integers less than $2A$ and prime to it

$$= 2A\left(\frac{a-1}{a}\right)\left(\frac{b-1}{b}\right)\left(\frac{c-1}{c}\right)\text{ \&c.}:$$

which, therefore, expresses the number of forms of prime numbers to the given modulus $4A$.

From this it appears, that the numbers having the least numbers of integers less than and prime to their halves, may be most advantageously employed as the moduli for forms expressive of prime numbers.

Ex. 1. Required the number of numbers less than 30, which are prime to it.

Here, $30 = 2.3.5$; therefore the number required will be

$$= 30\left(\frac{2-1}{2}\right)\left(\frac{3-1}{3}\right)\left(\frac{5-1}{5}\right) = 8; \text{ and the numbers are}$$

$$1,\ 7,\ 11,\ 13,\ 17,\ 19,\ 23,\ 29.$$

Ex. 2. Required the number of forms for prime numbers, when the modulus is 20.

Here $10 = 2.5$; whence we obtain the number required

$$= 10\left(\frac{2-1}{2}\right)\left(\frac{5-1}{5}\right) = 4:$$

and the forms themselves will be

$$20m \pm 1,\ 20m \pm 3,\ 20m \pm 7 \text{ and } 20m \pm 9.$$

71. To find a *perfect number*, or a number which is equal to the sum of all its divisors.

Let $N = a^p b$ be a perfect number, a and b being prime numbers: then, by article (384) the sum of its divisors

$$= \frac{a^{p+1}-1}{a-1} \times \frac{b^2-1}{b-1} = \frac{a^{p+1}-1}{a-1}(b+1):$$

$$\therefore\ a^p b = \frac{a^{p+1}-1}{a-1}(b+1) - a^p b, \text{ or } b = \frac{a^{p+1}-1}{a^{p+1}-2a^p+1}:$$

whence, in order that b may be an integer,

let $a^{p+1} - 2a^p + 1 = 1$, or $a = 2$: and $\therefore\ b = 2^{p+1} - 1$:

$$\therefore\ N = a^p b = 2^p(2^{p+1} - 1),$$

wherein p must be so assumed that $2^{p+1} - 1$ is a prime number.

If $p = 1$, we have $N = 2\ (2^2 - 1) = 6$:

if $p = 2$, ... $N = 2^2(2^3 - 1) = 28$:

if $p = 4$, ... $N = 2^4(2^5 - 1) = 496$: &c.

REVERSION AND INTERPOLATION OF SERIES.

72. Given $x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \&c. = y$, to express x in terms of y.

Assume $x = Ay + By^2 + Cy^3 + Dy^4 + \&c.$: then by substituting in the proposed equation, and equating the coefficients, we shall find that

$$x = y - \frac{1}{1.2}y^2 + \frac{1}{1.2.3}y^3 - \frac{1}{1.2.3.4}y^4 + \&c.$$

73. If $y = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \&c.$

$$\text{we have } y - 1 = x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \&c.:$$

and if $y - 1$ be denoted by z, and we assume

$$x = Az + Bz^2 + Cz^3 + Dz^4 + \&c.:$$

a similar process will give

$$x = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \frac{1}{4}z^4 + \&c.$$
$$= (y-1) - \frac{1}{2}(y-1)^2 + \frac{1}{3}(y-1)^3 - \frac{1}{4}(y-1)^4 - \&c.$$

74. If a series of equidistant terms be given, any intermediate term may be obtained by Interpolation, by means of the formula established in article (427): that is,

$$y = a + (x-1)\,\delta_1 + \frac{(x-1)(x-2)}{1.2}\,\delta_2 + \&c.$$

Ex. In the series $\frac{1}{50}, \frac{1}{51}, \frac{1}{52}, \frac{1}{53}, \frac{1}{54}$, &c., it will immediately appear that the middle term between $\frac{1}{52}$ and $\frac{1}{53}$ is $\frac{2}{105}$.

APPENDIX II.

MISCELLANEOUS EXAMPLES FOR PRACTICE.

FUNDAMENTAL OPERATIONS.

1. PROVE that $(a+b)^2c^2+(a-b)^2c^2=2(a^2+b^2)c^2$:

$$\text{and}\quad (a+b)^2c^2-(a-b)^2c^2=4abc^2.$$

2. Shew that $(x+y)^2+(x-y)^2+2z^2=2(x^2+y^2+z^2)$:

$$\text{and}\quad (x+y)^2+(x-y)^2+(2z)^2=2(x^2+y^2+2z^2).$$

3. Prove that $(ax+by)^2+(ay-bx)^2+c^2x^2+c^2y^2$

$$=(a^2+b^2+c^2)(x^2+y^2):\ \text{and that}$$

$$(ax+by)^2+(cx+dy)^2+(ay-bx)^2+(cy-dx)^2$$
$$=(a^2+b^2+c^2+d^2)(x^2+y^2).$$

4. Prove that

$$\{(ac+bd)x+(ad-bc)y\}^2+\{(ac+bd)y-(ad-bc)x\}^2$$
$$=(a^2+b^2)(c^2+d^2)(x^2+y^2).$$

5. Shew that $(a^2+b^2+1)(c^2+d^2+1)$ is always greater than $(ac+bd+1)^2$: and that $a^3-a^2b-ab^2+b^3$ is always positive, when a and b are unequal.

6. If $a=b\pm\delta$, prove that a^2+b^2 exceeds $2ab$ by δ^2: and that $a^6+a^4b^2+a^2b^4+b^6$ is generally greater than $(a^3+b^3)^2$.

7. If $x^2=a^2+b^2$, and $y^2=c^2+d^2$, it is required to shew that xy is greater than $ac+bd$ and $ad+bc$.

8. If a be greater than b, prove that a^m-b^m is less than $ma^{m-1}(a-b)$, and greater than $mb^{m-1}(a-b)$.

9. Shew that abc is greater than $(a+b-c)(a+c-b)(b+c-a)$, unless all the quantities are equal.

10. Prove that the sum or difference of any two quantities divided by their product, is equal to the sum or difference of their reciprocals.

11. Shew that the difference between the sum of the cubes of two quantities and the cube of their sum, is equal to three times their product multiplied by their sum.

12. Prove that $(1-x)(1+x)(1+x^2)(1+x^4)$. &c. to $n+1$ factors $=1-x^{2^n}$.

13. Prove that

$$4a^2b^2-(a^2+b^2-c^2)^2=s(s-2a)(s-2b)(s-2c), \text{ if } s=a+b+c.$$

14. If $r=\frac{1}{2}(a^2-b^2-c^2+d^2)$, it is required to prove that

$$(ad+bc)^2-r^2=\tfrac{1}{4}(a+b+c-d)(a+b+d-c)(a+c+d-b)(b+c+d-a).$$

15. If 1 be divided into any two parts, prove that the sums, formed by adding each part to the square of the other, are equal.

16. If 2 be divided into any two parts, the difference of their squares is always equal to twice the difference of the parts themselves.

17. Prove that $a^4-(b^2-c^2)^2+b^4-(a^2-c^2)^2+c^4-(a^2-b^2)^2$
$=(a+b+c)(a+b-c)(a+c-b)(b+c-a)$.

18. Shew that $(ab-cd)^2-(a+b-c-d)\times$
$\{ab(c+d)-cd(a+b)\}=(a-c)(a-d)(b-c)(b-d)$.

19. Prove that $a^{mn}-b^{mn}$ is divisible by both a^m-b^m and a^n-b^n: find the first and second terms and the last and last but one, and the number of terms in each quotient.

20. Shew that the quotient of

$$a^mb-ab^m-a^mc+ac^m+b^mc-bc^m \text{ by } (a-b)(a-c)$$

is $a^{m-2}(b-c)+a^{m-3}(b^2-c^2)+\text{\&c.}+a(b^{m-2}-c^{m-2})+(b^{m-1}-c^{m-1})$.

COMMON MEASURES.

1. The greatest common measure of $ax^2 - a^2x$ and $a^2x^2 + ab^2x$ is ax.

2. Of $x^3 - 5x^2 + 7x - 3$ and $x^2 + x - 12$ is $x - 3$.

3. Of $x^3 - 3x + 2$ and $x^3 + 4x^2 - 5$ is $x - 1$.

4. Of $x^3 + 1$ and $x^3 + mx^2 + mx + 1$ is $x + 1$.

5. Of $x^4 - 1$ and $x^3 + 3x^2 - 4$ is $x - 1$.

6. Of $x^3 - 8x^2 + 21x - 18$ and $3x^3 - 16x^2 + 21x$ is $x - 3$.

7. Of $7x^2 - 12x + 5$ and $2x^3 + x^2 - 8x + 5$ is $x - 1$.

8. Of $x^3 - 3x^2 - 10x + 24$ and $2ax^3 - 10ax^2 + 8ax$ is $x - 4$.

9. Of $x^3 - 4x^2 + 9x - 10$ and $x^3 + 2x^2 - 3x + 20$ is [illegible] $2x + 5$.

10. Of $x^3 - 19x^2 + 119x - 245$ and $3x^2 - 38x + 119$ is $x - 7$.

11. Of $2a^4 - 11a^2b^2 + 12b^4$ and $3a^5 - 48ab^4$ is $a^2 - 4b^2$.

12. Of $x^3 - 3a^2x - 2a^3$ and $x^4 - ax^3 + a^3x - 10a^4$ is $x - 2a$.

13. Of $2a^3 + 3a^2x - 9ax^2$ and $6a^3x - 17a^2x^2 + 14ax^3 - 3x^4$ is $2a - 3x$.

14. Of $x^4 - 4x^3 + 8x^2 - 16x + 16$ and $x^4 - 6x^3 + 13x^2 - 12x + 4$ is $(x - 2)^2$.

15. Of $48x^3 + 8x^2 + 31x + 15$ and $24x^3 + 22x^2 + 17x + 5$ is $12x + 5$.

16. Of $x^4 + ax^3 - 9a^2x^2 + 11a^3x - 4a^4$
and $x^4 - ax^3 - 3a^2x^2 + 5a^3x - 2a^4$ is $(x - a)^3$.

17. Of $3x^4 - 10x^3 + 9x^2 - 2x$ and $2x^4 - 7x^3 + 2x^2 + 8x$ is $x^2 - 2x$.

18. Of $2x^3 - 8x^2y + 16xy^2 - 16y^3$ and $8x^2 - 4xy - 24y^2$ is $2x - 4y$.

19. Of $4a^4 - 4a^2b^2 + 4ab^3 - b^4$
and $6a^4 + 4a^3b - 9a^2b^2 - 3ab^3 + 2b^4$ is $2a^2 + 2ab - b^2$.

20. Of $3x^2 - (4a + 2b)x + a^2 + 2ab$
and $x^3 - (2a + b)x^2 + (a + 2b)ax - a^2b$ is $x - a$.

21. Of $x^4 - 2a(a-b)x^2 + (a^2+b^2)(a-b)x - a^2b^2$ and $x^4 - (a-b)x^3 + (a-b)b^2x - b^4$ is $x^2 - (a-b)x + b^2$.

22. Of $45a^3b + 3a^2b^2 - 9ab^3 + 6b^4$ and $54a^2b - 24b^3$ is $9ab + 6b^2$.

23. Of $a^3 + a^2b + ab + b^2$ and $a^4 - b^2$ is $a^2 + b$.

24. Of $a^3 + (a+1)ay + y^2$ and $a^4 - a^2(y^2 - y) - y^3$ is $a^3 + a^2y + ay + y^2$.

25. Of $x^6 + x^2y - x^4y^2 - y^3$ and $x^4 - x^2y - x^2y^2 + y^3$ is $x^2 - y^2$.

26. Of $x^8 + a^2x^6 + ax^2 + a^3$ and $x^6 - a^4x^2 - ax^4 + a^5$ is $x^2 + a^2$.

27. Of $2a^2 + 6ab + 5ac + 4b^2 + 6bc + 2c^2$
and $9ac + 2a^2 - 10ab + 4c^2 + 2bc - 12b^2$ is $2a + 2b + c$.

28. Of $e^{2x}x^3 + e^{2x} - x^3 - 1$ and $e^{2x}x^2 + 2e^xx^2 - e^{2x} - 2e^x + x^2 - 1$ is $(x+1)(e^x+1)$.

29. Of $6x^3 + 4x^2y$, $2ax^4 - 8bx^2y^2$ and $4cx^5 + 12dx^4y$ is $2x^2$.

30. Of $a^3 + 5a^2x + 7ax^2 + 3x^3$, $a^3 + 3a^2x - ax^2 - 3x^3$ and $a^3 + a^2x - 5ax^2 + 3x^3$ is $a + 3x$.

COMMON MULTIPLES.

1. The least common multiple of axy and $a(xy - y^2)$ is $ax^2y - axy^2$: and of $ab + ad$ and $ab - ad$ is $ab^2 - ad^2$.

2. Of $x^3 + 1$ and $(x+1)^2$ is $x^4 + x^3 + x + 1$.

3. Of $x^3 - 7x^2 + 16x - 12$ and $3x^3 - 14x^2 + 16x$
is $3x^5 - 29x^4 + 104x^3 - 164x^2 + 96x$.

4. Of $12x^2 - 17ax + 6a^2$ and $9x^2 + 6ax - 8a^2$
is $36x^3 - 3ax^2 - 50a^2x + 24a^3$.

5. Of $a^3 + 2a^2b - ab^2 - 2b^3$ and $a^3 - 2a^2b - ab^2 + 2b^3$
is $a^4 - 5a^2b^2 + 4b^4$.

6. Of $a^2 - b^2$, $(a-b)^2$ and $a^3 + b^3$
is $a^5 - 2a^4b + a^3b^2 - a^2b^3 - 2ab^4 + b^5$.

EXAMPLES IN FRACTIONS.

1. Prove the correctness of the following results.

(1) $a+x+\dfrac{a^2-ax}{x}=\dfrac{a^2+x^2}{x}$, and $2a-x+\dfrac{(a-x)^2}{x}=\dfrac{a^2}{x}$.

(2) $a-x+\dfrac{4ax}{a-x}=\dfrac{(a+x)^2}{a-x}$, and $a+x-\dfrac{4ax}{a+x}=\dfrac{(a-x)^2}{a+x}$.

(3) $a^2+\left(\dfrac{2ax}{x^2-1}\right)^2=\left(\dfrac{x^2+1}{x^2-1}\right)^2a^2$: $\left(\dfrac{x^2+1}{2x}\right)^2-1=\left(\dfrac{x^2-1}{2x}\right)^2$.

(4) $$a^2-ax+x^2-\frac{2x^3}{a+x}=\frac{a^3-x^3}{a+x}:$$

$$(a-x)^2+\left(\frac{a^2+x^2}{a+x}\right)^2=\frac{2(a^4+x^4)}{(a+x)^2}.$$

(5) $$(a-x)^2+\frac{6a^2x+2x^3}{a-x}=\frac{(a+x)^3}{a-x}:$$

$$(a+x)^2-\frac{6a^2x+2x^3}{a+x}=\frac{(a-x)^3}{a+x}:$$

and $a^2-6ax+17x^2-\dfrac{16x^3(2a+x)}{(a+x)^2}=\dfrac{(a-x)^4}{(a+x)^2}$.

(6) $1-\dfrac{a^2+b^2-c^2}{2ab}=\dfrac{(a+c-b)(b+c-a)}{2ab}$:

and $b^2-\left(\dfrac{b^2+c^2-a^2}{2c}\right)^2=\dfrac{(a+b+c)(a+b-c)(a+c-b)(b+c-a)}{4c^2}$.

(7) $$1+\frac{a^2+b^2-c^2-d^2}{2(ab+cd)}=\frac{(a+b)^2-(c-d)^2}{2(ab+cd)}:$$

$$1-\frac{a^2+b^2-c^2-d^2}{2(ab+cd)}=\frac{(c+d)^2-(a-b)^2}{2(ab+cd)}.$$

(8) $$1-\left\{\frac{a^2+b^2-c^2-d^2}{2(ab+cd)}\right\}^2$$

$$=\frac{(a+b+c-d)(a+b+d-c)(a+c+d-b)(b+c+d-a)}{4(ab+cd)^2}.$$

(9) If $x^2 = \dfrac{(c^2 + d^2)\,ab - (a^2 + b^2)\,cd}{ab - cd}$, prove that

$$1 - \left\{\frac{a^2 + b^2 - x^2}{2ab}\right\}^2$$

$$= \frac{(a + b + c + d)(a + b - c - d)(a + c - b - d)(b + c - a - d)}{4\,(ab - cd)^2}.$$

2. It is required to establish the following results.

(1) $\dfrac{a^3 + x^3}{a - x} = a^2 + ax + x^2 + \dfrac{2x^3}{a - x}$, and $\dfrac{a^2}{a + x} = a - x + \dfrac{x^2}{a + x}$

(2) $$\frac{a^5}{a^2 - x^2} = a^3 + ax^2 + \frac{ax^4}{a^2 - x^2},$$

and $$\frac{a^5 x^5}{a^2 - x^2} = -\,a^5 x^3 - a^7 x + \frac{a^9 x}{a^2 - x^2}.$$

(3) $$\frac{x^2 + px + q}{x + a} = x - a + p + \frac{a^2 - pa + q}{x + a}.$$

3. Prove the following reductions to simplest forms.

(1) $\dfrac{a^2 + 2ab + b^2}{a^2 - b^2} = \dfrac{a + b}{a - b}$, and $\dfrac{a^3 - x^3}{(a - x)^2} = \dfrac{a^2 + ax + x^2}{a - x}$.

(2) $\dfrac{a^4 - b^4}{a^5 - a^3 b^2} = \dfrac{a^2 + b^2}{a^3}$, and $\dfrac{x^2 + (a - b)\,x - ab}{x^2 - (a + b)\,x + ab} = \dfrac{x + a}{x - a}$.

(3) $\dfrac{x^2 - 4x + 3}{x^2 + 2x - 3} = \dfrac{x - 1}{x + 1}$, and $\dfrac{x^2 + 2x - 3}{x^2 + 5x + 6} = \dfrac{x - 1}{x - 2}$.

(4) $$\frac{6a^3 - 6a^2 y + 2ay^2 - 2y^3}{12a^2 - 15ay + 3y^2} = \frac{6a^2 + 2y^2}{12a - 3y}.$$

(5) $$\frac{x^5 + 5bx^4 - b^2 x^2 - 5b^3 x}{x^4 + 3bx^3 - b^2 x - 3b^3} = \frac{x^2 + 5bx}{x + 3b}.$$

(6) $$\frac{4a^4 - 4a^2 b^2 + 4ab^3 - b^4}{6a^4 + 4a^3 b - 9a^2 b^2 - 3ab^3 + 2b^4} = \frac{2a^2 - 2ab + b^2}{3a^2 - ab - 2b^2}.$$

(7) $$\frac{27a^5b^2 - 18a^4b^2 - 9a^3b^2}{36a^6b^2 - 18a^5b^2 - 27a^4b^2 + 9a^3b^2} = \frac{3a+1}{4a^2+2a-1}.$$

(8) $$\frac{8a^2b^2 - 10ab^3 + 2b^4}{9a^4b - 9a^3b^2 + 3a^2b^3 - 3ab^4} = \frac{2b(4a-b)}{3a(3a^2+b^2)}.$$

(9) $$\frac{6x^5 + 15x^4y - 4x^3z^2 - 10x^2yz^2}{9x^3y - 27x^2yz - 6xyz^2 + 18yz^3} = \frac{x^2(2x+5y)}{3y(x-3z)}.$$

(10) $$\frac{a^2+b^2-c^2+2ab}{a^2-b^2-c^2+2bc} = \frac{a+b+c}{a-b+c}.$$

(11) $$\frac{a^4+a^3b+ab^3+b^4}{a^4+3a^3b+4a^2b^2+3ab^3+b^4} = \frac{a^2-ab+b^2}{a^2+ab+b^2}.$$

(12) $$\frac{a^4+a^3b+ab^3+b^4}{a^4-3a^3b+4a^2b^2-3ab^3+b^4} = \left(\frac{a+b}{a-b}\right)^2.$$

(13) $$\frac{a^2 - acx + (ac-b^2+bc)x^2 - bcx^3}{a^2 + abx + (ac-c^2+bc)x^2 + c^2x^3} = \frac{a-bx}{a+cx}.$$

(14) $$\frac{(ab-1)^2 + (a+b-2)(a+b-2ab)}{(ab+1)^2 - (a+b)^2} = \frac{(a-1)(b-1)}{(a+1)(b+1)}.$$

(15) $$\frac{(a^2-1)(b^2-1)ab + 4a^2b^2 - (a^2+b^2)(1+a^2b^2)}{(a^2+1)(b^2+1)ab - 4a^2b^2 - (a^2+b^2)(1+a^2b^2)} = \left(\frac{ab+1}{ab-1}\right)^2.$$

4. Verify the following reduced results.

(1) $$a - \frac{a^2}{a+b} = \frac{ab}{a+b}, \text{ and } \frac{a^2+b^2}{a^2-b^2} - \frac{a-b}{a+b} = \frac{2ab}{a^2-b^2}.$$

(2) $$\frac{1}{a^m-1} - \frac{1}{a^m+1} = \frac{2}{a^{2m}-1}, \text{ and } \frac{x}{a^x-1} + \frac{x}{a^{-x}-1} = -x.$$

(3) $$\frac{a}{a+b} + \frac{b}{a-b} = \frac{a^2+b^2}{a^2-b^2} = \frac{a}{a-b} - \frac{b}{a+b}.$$

(4) $$\frac{a}{a+c} - \frac{b}{b+c} = \frac{(a-b)c}{(a+c)(b+c)} = \frac{c}{b+c} - \frac{c}{a+c}.$$

(5) $$2+\frac{a^2+b^2}{a^2-b^2}-\frac{a-b}{a+b}=\frac{2\,(a^2+ab-b^2)}{a^2-b^2}.$$

(6) $$\frac{b}{d}+\frac{ad-bc}{d(c+dx)}=\frac{a+bx}{c+dx},\ \text{and}\ \frac{a}{c}+\frac{(ad-bc)x}{c\,(c-dx)}=\frac{a-bx}{c-dx}.$$

(7) $$\frac{1}{4a^2\,(x-a)}+\frac{1}{4a^2\,(x+a)}-\frac{x}{2a^2\,(x^2+a^2)}=\frac{x}{x^4-a^4}.$$

(8) $$\frac{1}{x-1}-\frac{1}{2x+2}-\frac{x+3}{2x^2+2}=\frac{x+3}{x^4-1}.$$

(9) $$\frac{2}{x+a}+\frac{3a}{(x+a)^2}+\frac{3a-2x}{x^2-2ax+3a^2}=\frac{18a^3}{x^4+4a^3x+3a^4}.$$

(10) $$\frac{2}{x}-\frac{1}{a+x}+\frac{1}{a-x}=\frac{2a^2}{x\,(a^2-x^2)}.$$

(11) $$\frac{1}{2\,(x-1)}-\frac{4}{x-2}+\frac{9}{2\,(x-3)}=\frac{x^2}{(x-1)(x-2)(x-3)}.$$

(12) $$\frac{1}{8\,(x-1)}-\frac{1}{4(x-3)}+\frac{1}{8(x-5)}=\frac{1}{(x-1)(x-3)(x-5)}.$$

(13) $$\frac{3}{(1+x)^2}-\frac{1}{1+x}-\frac{1}{1-x}=\frac{1-5x}{(1-x)\,(1+x)^2}.$$

(14) $$\frac{1}{3}\left(\frac{1-2x}{x^2-x+1}\right)+\frac{1}{2}\left(\frac{1+x}{x^2+1}\right)+\frac{1}{6}\left(\frac{1}{x+1}\right)=\frac{1}{(x^2+1)(x^3+1)}.$$

(15) $$\frac{1}{x}-\frac{1}{(x+1)^2}-\frac{1}{x+1}+\frac{x}{1+x+x^2}=\frac{1}{x(1+x)^2(1+x+x^2)}.$$

(16) $$\frac{a^2}{x+a}+\frac{b^2-2ab}{x+b}+\frac{(a-b)\,b^2}{(x+b)^2}=\frac{(a-b)^2\,x^2}{(x+a)\,(x+b)^2}.$$

(17) $$\frac{x}{x-3}-\frac{x-3}{x}+\frac{x}{x+3}-\frac{x+3}{x}=\frac{18}{x^2-9}.$$

(18) $$\frac{1}{x^3}+\frac{1}{x^2}-\frac{1}{x}-\frac{1}{(x^2+1)^2}+\frac{x-1}{x^2+1}=\frac{x^2+x+1}{x^3\,(x^2+1)^2}.$$

(19) $$\frac{a+b-c}{a-b+c}-\frac{a-b+c}{a+b-c}-\frac{4\,(b-c)^2}{a^2-(b-c)^2}=\frac{4\,(b-c)}{a+b-c}.$$

(20) $$\frac{1}{(a-b)(a-c)(x+a)} - \frac{1}{(a-b)(b-c)(x+b)}$$
$$+ \frac{1}{(a-c)(b-c)(x+c)} = \frac{1}{(x+a)(x+b)(x+c)}.$$

(21) $$\frac{a^2}{(a-b)(a-c)(x+a)} - \frac{b^2}{(a-b)(b-c)(x+b)}$$
$$+ \frac{c^2}{(a-c)(b-c)(x+c)} = \frac{x^2}{(x+a)(x+b)(x+c)}.$$

5. It is required to establish the following results.

(1) $\frac{a}{bx} \times \frac{cx}{d} = \frac{ac}{bd}$, and $\frac{5ax}{bcy}\left(\frac{xy+y^2}{x^2-xy}\right) = \frac{5a(x+y)}{bc(x-y)}$.

(2) $$\frac{a^2+ax+x^2}{a^3-a^2x+ax^2-x^3} \times \frac{a^2-ax+x^2}{a+x} = \frac{a^4+a^2x^2+x^4}{a^4-x^4}.$$

(3) $$\frac{x^2-9x+20}{x^2-6x} \times \frac{x^2-13x+42}{x^2-5x} = \frac{x^2-11x+28}{x^2}.$$

(4) $$\frac{4ax}{3by} \times \frac{a^2-x^2}{c^2-x^2} \times \frac{bc+bx}{a^2-ax} = \frac{4x(a+x)}{3y(c-x)}.$$

(5) $$\frac{a^2-b^2}{x+y} \times \frac{x^2-y^2}{a-b} \times \frac{a^2}{(x-y)^2} = \frac{a^2(a+b)}{x-y}.$$

(6) $$\left(a+\frac{ax}{a-x}\right) \times \left(a-\frac{ax}{a+x}\right) \times \frac{a^2-x^2}{a^2+x^2} = \frac{a^4}{a^2+x^2}.$$

(7) $$\left(\frac{a}{a-b}+\frac{b}{a+b}\right)\left(\frac{a}{a-b}-\frac{b}{a+b}\right)\left(\frac{a+b}{a-b}\right)^2$$
$$= \frac{a^3(a+2b)-b^3(b-2a)}{(a-b)^4}.$$

6. Multiply $15a^{-6}b^2-7a^{-5}b^4+6a^{-4}b^6$ by $8a^{-2}b^2-3a^{-1}b^4$.

Answer: $120a^{-8}b^4-101a^{-7}b^6+69a^{-6}b^8-18a^{-5}b^{10}$.

(5) $$2+\frac{a^2+b^2}{a^2-b^2}-\frac{a-b}{a+b}=\frac{2\,(a^2+ab-b^2)}{a^2-b^2}.$$

(6) $$\frac{b}{d}+\frac{ad-bc}{d(c+dx)}=\frac{a+bx}{c+dx},\ \text{and}\ \frac{a}{c}+\frac{(ad-bc)x}{c\,(c-dx)}=\frac{a-bx}{c-dx}.$$

(7) $$\frac{1}{4a^2\,(x-a)}+\frac{1}{4a^2\,(x+a)}-\frac{x}{2a^2\,(x^2+a^2)}=\frac{x}{x^4-a^4}.$$

(8) $$\frac{1}{x-1}-\frac{1}{2x+2}-\frac{x+3}{2x^2+2}=\frac{x+3}{x^4-1}.$$

(9) $$\frac{2}{x+a}+\frac{3a}{(x+a)^2}+\frac{3a-2x}{x^2-2ax+3a^2}=\frac{18a^3}{x^4+4a^3x+3a^4}.$$

(10) $$\frac{2}{x}-\frac{1}{a+x}+\frac{1}{a-x}=\frac{2a^2}{x\,(a^2-x^2)}.$$

(11) $$\frac{1}{2\,(x-1)}-\frac{4}{x-2}+\frac{9}{2\,(x-3)}=\frac{x^2}{(x-1)(x-2)(x-3)}.$$

(12) $$\frac{1}{8\,(x-1)}-\frac{1}{4(x-3)}+\frac{1}{8(x-5)}=\frac{1}{(x-1)(x-3)(x-5)}.$$

(13) $$\frac{3}{(1+x)^2}-\frac{1}{1+x}-\frac{1}{1-x}=\frac{1-5x}{(1-x)\,(1+x)^2}.$$

(14) $$\frac{1}{3}\left(\frac{1-2x}{x^2-x+1}\right)+\frac{1}{2}\left(\frac{1+x}{x^2+1}\right)+\frac{1}{6}\left(\frac{1}{x+1}\right)=\frac{1}{(x^2+1)(x^3+1)}.$$

(15) $$\frac{1}{x}-\frac{1}{(x+1)^2}-\frac{\sim}{x+1}+\frac{x}{1+x+x^2}=\frac{1}{x\,(1+x)^2\,(1+x+x^2)}.$$

(16) $$\frac{a^2}{x+a}+\frac{b^2-2ab}{x+b}+\frac{(a-b)\,b^2}{(x+b)^2}=\frac{(a-b)^2\,x^2}{(x+a)\,(x+b)^2}.$$

(17) $$\frac{x}{x-3}-\frac{x-3}{x}+\frac{x}{x+3}-\frac{x+3}{x}=\frac{18}{x^2-9}.$$

(18) $$\frac{1}{x^3}+\frac{1}{x^2}-\frac{1}{x}-\frac{1}{(x^2+1)^2}+\frac{x-1}{x^2+1}=\frac{x^2+x+1}{x^3\,(x^2+1)^2}.$$

(19) $$\frac{a+b-c}{a-b+c}-\frac{a-b+c}{a+b-c}-\frac{4\,(b-c)^2}{a^2-(b-c)^2}=\frac{4\,(b-c)}{a+b-c}.$$

12. Divide $\frac{a^4}{b^6} - \frac{4c^6d^8}{b^{10}} + \frac{14c^5d^4}{a^4b^8} - \frac{49c^4}{4a^8b^6}$ by

$$\frac{a^2}{b^3} - \frac{2c^3d^4}{b^5} + \frac{7c^2}{2a^4b^3}.$$

Answer: $\frac{a^2}{b^3} + \frac{2c^3d^4}{b^5} - \frac{7c^2}{2a^4b^3}$.

13. Divide $-2a^{-8}x^5 + 17a^{-4}x^6 - 5x^7 - 24a^4x^8$ by

$$2a^{-3}x^3 - 3ax^4.$$

Answer: $-a^{-5}x^2 + 7a^{-1}x^3 + 8a^3x^4$.

14. The cube of $\frac{x}{a} + \frac{a}{x} = \frac{x^3}{a^3} + \frac{a^3}{x^3} + 3\left(\frac{x}{a} + \frac{a}{x}\right)$: and of

$$\frac{x}{y} - 1 - \frac{y}{x} = \frac{x^3}{y^3} - \frac{y^3}{x^3} - 3\left(\frac{x^2}{y^2} + \frac{y^2}{x^2}\right) + 5.$$

15. The square root of $\frac{a^2}{b^2} + \frac{b^2}{a^2} - 2 = \frac{a}{b} - \frac{b}{a}$.

16. Of $x^2 + \frac{1}{x^2} + 2\left(x - \frac{1}{x}\right) - 1 = x - \frac{1}{x} + 1$.

17. Of $\frac{a^2}{x^2} - \frac{4a}{3b} + \frac{4x^2}{9b^2} = \frac{a}{x} - \frac{2x}{3b}$.

18. Of $\frac{x}{a}\left(\frac{x}{a} - 2\right) + \frac{b}{x}\left(\frac{b}{x} - 2\right) + \frac{2b+a}{a} = \frac{x}{a} - 1 + \frac{b}{x}$.

19. Of $\frac{a^4}{b^2} + \frac{c^2}{a^4} + \frac{bc^2}{a^2}\left(b + \frac{2}{a}\right) + 2c\left(a + \frac{1}{b}\right) = \frac{a^2}{b} + \frac{bc}{a} + \frac{c}{a^2}$.

20. Of $\frac{4x^2}{49y^2} - \frac{20x}{7y} + \frac{178}{7} - \frac{15y}{2x} + \frac{9y^2}{16x^2} = \frac{2x}{7y} - 5 + \frac{3y}{4x}$.

21. Prove the following symbolical equalities.

(1) $\frac{a}{x+1} = \frac{a}{x} - \frac{a}{x^2} + \frac{a}{x^3} - \frac{a}{x^4} + \&c.$

(2) $\dfrac{x+a}{x-b} = 1 + \dfrac{a+b}{x} + \dfrac{b(a+b)}{x^2} + \dfrac{b^2(a+b)}{x^3} + \text{\&c.}$

(3) $\dfrac{a-x}{b+x} = \dfrac{a}{b} - \dfrac{a-b}{b^2}x + \dfrac{a-b}{b^3}x^2 - \dfrac{a-b}{b^4}x^3 + \text{\&c.}$

22. If $\dfrac{a^2(b-c)}{a-d} = \dfrac{b^2(a-c)}{b-d}$, then $\dfrac{1}{a} + \dfrac{1}{b} = \dfrac{1}{c} + \dfrac{1}{d}$.

23. Find the value of $\dfrac{x+2a}{x-2a} + \dfrac{x+2b}{x-2b}$, when $x = \dfrac{4ab}{a+b}$.

Answer: 2.

24. Prove that $a^x + a^y \div a^{-x} + a^{-y} = a^{x+y}$.

25. Establish the following formula:

$$\frac{1}{1+x^{m-n}+x^{m-p}} + \frac{1}{1+x^{n-m}+x^{n-p}} + \frac{1}{1+x^{p-m}+x^{p-n}} = 1.$$

EXAMPLES IN SURDS.

1. Prove that

$$(a+x)\sqrt{\frac{a-x}{a+x}} = \sqrt{a^2-x^2},\quad (a+x)\sqrt{\frac{1}{a^2-x^2}} = \sqrt{\frac{a+x}{a-x}},$$

$$\text{and}\quad (a-x)\sqrt{\frac{1}{a^2-x^2}} = \sqrt{\frac{a-x}{a+x}}.$$

2. $\sqrt{3a^2x + 6abx + 3b^2x} = (a+b)\sqrt{3x}$.

3. $\sqrt{a^3 - 3ax^2 + 2x^3} = (a-x)\sqrt{a+2x}$.

4. $\sqrt{a^3x^3 - 3ab^2x - 2b^3} = (ax+b)\sqrt{ax-2b}$.

5. $\sqrt[3]{a^4 - 6a^2x^2 - 8ax^3 - 3x^4} = (a+x)\sqrt[3]{a-3x}$.

6. $\sqrt[3]{4a^3x + 12a^2x^2 + 12ax^3 + 4x^4} = (a+x)\sqrt[3]{4x}$.

7. Establish the following results.

(1) $\sqrt[4]{16a^4b^2c^4} - a\sqrt{bc^2} = ac\sqrt{b}$: $\sqrt{a^3b} + \sqrt{ab^3}$ $= (a+b)\sqrt{ab}$, and $\sqrt[3]{a^4x} - \sqrt[3]{ax^4} = (a-x)\sqrt[3]{ax}$.

(2) $\sqrt{48a^4x}+\sqrt{27a^2x^3}+\sqrt{12x^5}=(4a^2+3ax+2x^2)\sqrt{3x}$.

(3) $x\sqrt{12a^2x}+2a\sqrt{27x^3}+3\sqrt{48a^2x^3}=20ax\sqrt{3x}$.

(4) $\sqrt{45x^3}-\sqrt{80x^3}+\sqrt{5a^2x}=(a-x)\sqrt{5x}$.

(5) $$\left(\frac{16a^4x}{3b^4}\right)^{\frac{1}{3}}+\left(\frac{2axy^3}{81bc^3}\right)^{3}-\left(\frac{ax^4}{12b^4}\right)^{\frac{1}{3}}$$

$$=\left(\frac{2a}{b}+\frac{y}{3c}-\frac{x}{2b}\right)\left(\frac{2ax}{3b}\right)^{\frac{1}{3}}.$$

(6) $\left\{a+(ab)^{\frac{1}{2}}+b\right\}(a^{\frac{1}{2}}-b^{\frac{1}{2}})=a^{\frac{3}{2}}-b^{\frac{3}{2}}$.

(7) $\left\{x^{\frac{1}{2}}+(xy)^{\frac{1}{4}}+y^{\frac{1}{2}}\right\}\left\{x^{\frac{1}{2}}-(xy)^{\frac{1}{4}}+y^{\frac{1}{2}}\right\}=x+(xy)^{\frac{1}{2}}+y$.

(8) $(a+b^{\frac{1}{2}}-d)(a-b^{\frac{1}{2}})=a^2-ad-b+db^{\frac{1}{2}}$.

(9) $(ax^{\frac{1}{2}}+by^{\frac{1}{2}})(ax^{\frac{1}{2}}-by^{\frac{1}{2}})=a^2x-b^2y$.

(10) $(a^2+ab^{\frac{1}{3}}+b^{\frac{2}{3}})(a-b^{\frac{1}{3}})=a^3-b$.

(11) $(x^{\frac{1}{2}}+x^{\frac{1}{4}}y^{\frac{1}{4}}+y^{\frac{1}{2}})(x^{\frac{1}{4}}-y^{\frac{1}{4}})=x^{\frac{3}{4}}-y^{\frac{3}{4}}$.

(12) $(x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}})(x^{\frac{1}{3}}+y^{\frac{1}{3}})=x+y$.

(13) $(a^{\frac{3}{2}}-ax^{\frac{3}{4}}+a^{\frac{1}{2}}x^{\frac{3}{2}}-x^{\frac{9}{4}})(a^{\frac{1}{2}}+x^{\frac{3}{4}})=a^2-x^3$.

(14) $(x^{\frac{1}{2}}+y^{\frac{1}{2}})(x^{-\frac{1}{2}}+y^{-\frac{1}{2}})=x^{\frac{1}{2}}y^{-\frac{1}{2}}+2+x^{-\frac{1}{2}}y^{\frac{1}{2}}$.

(15) $$(a^2+3x^2)(a^2-x^2)^{\frac{1}{2}}-\frac{(a^2+x^2)x^2}{(a^2-x^2)^{\frac{1}{2}}}=\frac{a^4+a^2x^2-4x^4}{(a^2-x^2)^{\frac{1}{2}}}.$$

(16) $$4b(ax+bx^2)^{\frac{1}{2}}-(a+2bx)^2(ax+bx^2)^{-\frac{1}{2}}=-\frac{a^2}{(ax+bx^2)^{\frac{1}{2}}}.$$

(17) $$4c(a+bx+cx^2)^{\frac{1}{2}}-\frac{(b+2cx)^2}{(a+bx+cx^2)^{\frac{1}{2}}}=\frac{4ac-b^2}{(a+bx+cx^2)^{\frac{1}{2}}}.$$

(18) $$\frac{1}{a-\sqrt{a^2-x^2}}-\frac{1}{a+\sqrt{a^2-x^2}}=\frac{2\sqrt{a^2-x^2}}{x^2}.$$

(19) $$\frac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}}-\frac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}}=4x\sqrt{x^2-1}.$$

(20) $(x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y)(x^{-1} + x^{-\frac{1}{2}}y^{-\frac{1}{2}} + y^{-1})$
$= xy^{-1} + 2x^{\frac{1}{2}}y^{-\frac{1}{2}} + 3 + 2x^{-\frac{1}{2}}y^{\frac{1}{2}} + x^{-1}y.$

(21) $11x\sqrt{a^2 - ax} \div x^2\sqrt{a^2x - ax^2} = \dfrac{11}{x\sqrt{x}}.$

(22) $(a - b) \div (a^{\frac{1}{3}} - b^{\frac{1}{3}}) = a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}.$

(23) $\{ab^{\frac{1}{2}} + b(ac)^{\frac{1}{2}} - (abc)^{\frac{1}{2}} - bc\} \div \{a^{\frac{1}{2}} + (bc)^{\frac{1}{2}}\}$
$= (ab)^{\frac{1}{2}} - (bc)^{\frac{1}{2}}.$

(24) $(x^{\frac{3}{2}} + 2xy^{\frac{1}{2}} + 2x^{\frac{1}{2}}y + y^{\frac{3}{2}}) \div (x^{\frac{1}{2}} + y^{\frac{1}{2}}) = x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y.$

(25) $x^{\frac{5}{2}} - x^2y^{\frac{1}{2}} + x^{\frac{3}{2}}y - xy^{\frac{3}{2}} + x^{\frac{1}{2}}y^2 - y^{\frac{5}{2}}) \div (x^2 + xy + y^2)$
$= x^{\frac{1}{2}} - y^{\frac{1}{2}}.$

(26) $(a^{\frac{7}{3}} - a^2b^{-\frac{2}{3}} - a^{\frac{1}{3}}b + b^{\frac{1}{3}}) \div (a^{\frac{1}{3}} - b^{-\frac{2}{3}}) = a^2 - b.$

(27) $(a^{\frac{1}{2}}x^{-\frac{1}{3}} + a^{-\frac{1}{2}}x^{\frac{1}{3}} + a^{\frac{1}{2}}y^{-\frac{1}{4}} + a^{-\frac{1}{2}}y^{\frac{1}{4}} + x^{\frac{1}{3}}y^{-\frac{1}{4}} + x^{-\frac{1}{3}}y^{\frac{1}{4}} + 3)$
$\div (a^{-\frac{1}{2}} + x^{-\frac{1}{3}} + y^{-\frac{1}{4}}) = a^{\frac{1}{2}} + x^{\frac{1}{3}} + y^{\frac{1}{4}}.$

8. The square of $2\sqrt{a} + 3\sqrt{x} = 4a + 9x + 12\sqrt{ax}$: and the cube of $x^{\frac{1}{3}} - a^{\frac{1}{2}} = x - 3a^{\frac{1}{2}}x^{\frac{2}{3}} + 3ax^{\frac{1}{3}} - a^{\frac{3}{2}}.$

9. Shew that the cube of $x^{\frac{1}{3}} - x^{-\frac{1}{3}}y^{\frac{2}{3}} - y^{\frac{1}{3}}$
is $x - 3x^{\frac{2}{3}}y^{\frac{1}{3}} + 5y - 3x^{-\frac{2}{3}}y^{\frac{5}{3}} + x^{-1}y^2.$

10. The square of $\sqrt{1 - x + x^2} + \sqrt{1 + x - x^2}$ is
$2 + 2\sqrt{1 - x^2 + 2x^3 - x^4}.$

11. The square root of $ax^2 + by^2 - 2xy\sqrt{ab}$
is $x\sqrt{a} - y\sqrt{b}$:

and of $\dfrac{x^2}{y^2} + \dfrac{y^2}{x^2} - \left(\dfrac{x}{y} + \dfrac{y}{x}\right)\sqrt{2} + 2\frac{1}{2}$ is $\dfrac{x}{y} - \dfrac{1}{\sqrt{2}} + \dfrac{y}{x}.$

12. Of $1 + \dfrac{41a}{16} - \dfrac{3(1 + a)}{2}\sqrt{a} + a^2$ is $1 - \frac{3}{4}\sqrt{a} + a$:

and of $4a^2 + bx - y^2 - 2\sqrt{4a^2bx - bxy^2}$ si $\sqrt{4a^2 - y^2} - \sqrt{bx}.$

13. The square root of $a+b+c+2\{\sqrt{ab}+\sqrt{ac}+\sqrt{bc}\}$ is $\sqrt{a}+\sqrt{b}+\sqrt{c}$.

14. The square root of $a+b+c+d+2\sqrt{ad+bd+cd}$ is $\sqrt{a+b+c}+\sqrt{d}$: and of $a+b+c+d+2\sqrt{ac+bc+ad+bd}$ is $\sqrt{a+b}+\sqrt{c+d}$.

15. Extract the square roots of the following surds.

(1) Of $7+4\sqrt{3}$, the root is $2+\sqrt{3}$.

(2) Of $11-6\sqrt{2}$, ... $3-\sqrt{2}$.

(3) Of $32+10\sqrt{7}$, ... $5+\sqrt{7}$.

(4) Of $28-5\sqrt{12}$, ... $5-\sqrt{3}$.

(5) Of $36+10\sqrt{11}$, ... $5+\sqrt{11}$.

(6) Of $5+\sqrt{24}$, ... $\sqrt{3}+\sqrt{2}$.

(7) Of $12+2\sqrt{35}$, ... $\sqrt{5}+\sqrt{7}$.

(8) Of $87-12\sqrt{42}$, ... $3\sqrt{7}-2\sqrt{6}$.

(9) Of $4\frac{1}{2}+2\sqrt{2}$, ... $\frac{1}{2}(4+\sqrt{2})$.

(10) Of $3\frac{1}{2}-\sqrt{10}$, ... $\frac{1}{2}(2-\sqrt{10})$.

(11) Of $1\frac{7}{9}-2\frac{2}{3}\sqrt{\frac{1}{3}}$, ... $\frac{2}{3}(\sqrt{3}-1)$.

(12) Of $\sqrt{18}+4$, ... $\sqrt[4]{8}+\sqrt[4]{2}$.

(13) Of $\sqrt{27}-2\sqrt{6}$, ... $\sqrt[4]{12}-\sqrt[4]{3}$.

(14) Of $3\sqrt{5}+2\sqrt{10}$, ... $\sqrt[4]{20}+\sqrt[4]{5}$.

(15) Of $3\sqrt{6}-4\sqrt{3}$, ... $\sqrt[4]{24}-\sqrt[4]{6}$.

16. Extract the square root of $2a+2\sqrt{a^2-x^2}$.

Answer: $\sqrt{a+x}+\sqrt{a-x}$.

17. Extract the square root of $ax - 2a\sqrt{ax - a^2}$.

Answer: $\sqrt{ax - a^2} - a$.

18. Extract the square root of $\frac{1}{4}a^2 + \frac{1}{2}x\sqrt{a^2 - x^2}$.

Answer: $\frac{1}{2}x + \frac{1}{2}\sqrt{a^2 - x^2}$.

19. Extract the square root of $2 + 2(1 - x)\sqrt{1 + 2x - x^2}$.

Answer: $\sqrt{1 + 2x - x^2} + 1 - x$.

20. Extract the fourth root of $17 + 12\sqrt{2}$.

Answer: $1 + \sqrt{2}$.

21. Extract the fourth root of $14 - 8\sqrt{3}$.

Answer: $\sqrt[4]{\frac{9}{2}} - \sqrt[4]{\frac{1}{2}}$.

22. Extract the cube root of $10 + 6\sqrt{3}$.

Answer: $1 + \sqrt{3}$.

23. Extract the cube root of $38 - 17\sqrt{5}$.

Answer: $2 - \sqrt{5}$.

24. Prove that $\sqrt{3 + \sqrt{5}} + \sqrt{3 - \sqrt{5}} = \sqrt{10}$:

and $\sqrt{3 + \sqrt{5}} - \sqrt{3 - \sqrt{5}} = \sqrt{2}$.

25. Find the sum and difference of the quantities,

$$\frac{a + b\sqrt{-1}}{a - b\sqrt{-1}}, \text{ and } \frac{a - b\sqrt{-1}}{a + b\sqrt{-1}}.$$

Answers: $\dfrac{2(a^2 - b^2)}{a^2 + b^2}$, and $\dfrac{4ab\sqrt{-1}}{a^2 + b^2}$.

26. Required the sum and product of the quantities,

$$\frac{a + b\sqrt{-1}}{x + y\sqrt{-1}}, \text{ and } \frac{a - b\sqrt{-1}}{x - y\sqrt{-1}}.$$

Answers: $\dfrac{2(ax + by)}{x^2 + y^2}$, and $\dfrac{a^2 + b^2}{x^2 + y^2}$.

27. Extract the following square roots.

(1) Of $-1+4\sqrt{-5}$, the root is $2+\sqrt{-5}$.

(2) Of $-3+4\sqrt{-7}$, ... $2+\sqrt{-7}$.

(3) Of $19-10\sqrt{-6}$, ... $5-\sqrt{-6}$.

(4) Of $31-12\sqrt{-5}$, ... $6-\sqrt{-5}$.

(5) Of $24\sqrt{-1}-7$, ... $3+4\sqrt{-1}$.

28. Extract the square root of $2a+2\sqrt{a^2+b^2}$.

Answer: $\sqrt{a+b\sqrt{-1}}+\sqrt{a-b\sqrt{-1}}$.

29. Find the square root of the quantity

$$a^2+b^2-2\sqrt{a^2b^2+(a^2-b^2)\,x^2-x^4}.$$

Answer: $\sqrt{x^2+b^2}-\sqrt{-1}\sqrt{x^2-a^2}$.

30. Prove the truth of the following results:

$$\frac{1+\sqrt{-1}}{1-\sqrt{-1}}=\sqrt{-1}:\qquad \frac{5-\sqrt{-2}}{1+\sqrt{-2}}=1-2\sqrt{-2}:$$

$$\frac{1}{1+2\sqrt{-1}}=\frac{1}{5}(1-2\sqrt{-1}):\qquad \frac{21}{3-2\sqrt{-3}}=3+2\sqrt{-3}.$$

31. Find the continued product of $x-1-\sqrt{-2}$, $x-1+\sqrt{-2}$, $x-2+\sqrt{-3}$, and $x-2-\sqrt{-3}$.

Answer: $x^4-6x^3+18x^2-26x+21$.

32. Required the continued product of $x+a$, $x-a$, $x-\frac{1}{2}a(1+\sqrt{-3})$, $x-\frac{1}{2}a(1-\sqrt{-3})$, $x+\frac{1}{2}a(1+\sqrt{-3})$, and $x+\frac{1}{2}a(1-\sqrt{-3})$.

Answer: x^6-a^6.

33. If $u=\frac{1}{2}\left(x+\frac{1}{x}\right)$, and $v=\frac{1}{2}\left(y+\frac{1}{y}\right)$, prove that

$$uv+\sqrt{1-u^2}\sqrt{1-v^2}=\frac{1}{2}\left(\frac{x}{y}+\frac{y}{x}\right):$$

$$uv-\sqrt{1-u^2}\sqrt{1-v^2}=\frac{1}{2}\left(xy+\frac{1}{xy}\right).$$

SIMPLE EQUATIONS.

Solve the following simple equations.

(1) $7x - 3 = 5x + 13.$ $\quad x = 8.$

(2) $3x + 5 = 10x - 16.$ $\quad x = 3.$

(3) $2x + 11 = 7x - 14.$ $\quad x = 5.$

(4) $15x - 24 = 20 + \frac{1}{3}x.$ $\quad x = 3.$

(5) $x - \frac{x}{2} + \frac{x}{3} - \frac{x}{4} = 7.$ $\quad x = 12.$

(6) $\frac{x+1}{2} + \frac{x+2}{3} = 14 + \frac{5-x}{4}.$ $\quad x = 13.$

(7) $\frac{x}{2} - \frac{5x+4}{3} = \frac{4x-9}{3}.$ $\quad x = \frac{2}{3}.$

(8) $\frac{x+6}{4} + \frac{16-3x}{12} = \frac{x+9}{6}.$ $\quad x = 8.$

(9) $\frac{x+1}{2} + \frac{x+2}{3} = 16 - \frac{5x+1}{4}.$ $\quad x = 7.$

(10) $\frac{x-7}{11} - \frac{3x-5}{7} + \frac{125}{77} = 2x - 17.$ $\quad x = 8.$

(11) $\frac{5x-7}{3} - \frac{3x-2}{7} = \frac{x-5}{4}.$ $\quad x = \frac{67}{83}.$

(12) $\frac{x+3}{2} - \frac{11-x}{5} = \frac{3x-1}{20} + 3\frac{1}{5}.$ $\quad x = 7.$

(13) $x - \frac{x-2}{3} = 5\frac{3}{4} - \frac{x+10}{5} + \frac{x}{4}.$ $\quad x = 5.$

(14) $\frac{9x+7}{2} - \left(x - \frac{x-2}{7}\right) = 36.$ $\quad x = 9.$

(15) $\frac{3x+7}{14} - \frac{2x-7}{21} + 2\frac{3}{4} = \frac{x-4}{4}.$ $\quad x = 35.$

(16) $\frac{2x+1}{29} - \frac{402-3x}{12} = 9 - \frac{471-6x}{2}.$ $\quad x = 72.$

(17) $\frac{4x-21}{7}+7\frac{3}{4}+\frac{7x-28}{3}=x+3\frac{3}{4}-\frac{9-7x}{8}$. $x=7$.

(18) $\frac{x-1\frac{25}{26}}{2}-\frac{2-6x}{13}=x-\frac{5x-\frac{1}{4}(10-3x)}{39}$. $x=11$.

(19) $\frac{\frac{3}{4}x-2}{8}+19=\frac{\frac{1}{2}x+1-\frac{1}{5}(x-4)+x}{11}$. $x=24$.

(20) $\frac{17-3x}{5}-\frac{2x+1}{1\frac{1}{2}}=\frac{29-11x}{3}$. $x=4$.

(21) $.15x+.2-.875x+.375=.0625x-1$. $x=2$.

(22) $\frac{6x+13}{15}-\frac{3x+5}{5x-25}=\frac{2x}{5}$. $x=20$.

(23) $\frac{4x+3}{9}+\frac{7x-29}{5x-12}=\frac{8x+19}{18}$. $x=6$.

(24) $\frac{2x+8\frac{1}{2}}{9}-\frac{13x-2}{17x-32}+\frac{x}{3}=\frac{7x}{12}-\frac{x+16}{36}$. $x=4$.

(25) $\frac{41-35x}{105}-\frac{7-2x^2}{14(x-1)}=\frac{1+3x}{21}-\frac{2x-2\frac{1}{5}}{6}$. $x=4$.

(26) $\frac{6x-7\frac{1}{3}}{13-2x}+2x+\frac{1+16x}{24}=4\frac{5}{12}-\frac{12\frac{5}{8}-8x}{3}$. $x=1\frac{1}{2}$.

(27) $\frac{25-\frac{1}{3}x}{x+1}+\frac{16x+4\frac{1}{5}}{3x+2}=5+\frac{23}{x+1}$. $x=3\frac{3}{8}$.

(28) $\frac{3-2x}{1-2x}-\frac{5-2x}{7-2x}=1-\frac{4x^2-2}{7-16x+4x^2}$. $x=-\frac{7}{8}$.

(29) $\sqrt{x+9}=1+\sqrt{x}$. $x=16$.

(30) $\sqrt{x+11}-\sqrt{x}=1$. $x=25$.

(31) $\sqrt{x+13}+\sqrt{x}=13$. $x=36$.

(32) $(\sqrt{x}+28)(\sqrt{x}+6)=(\sqrt{x}+38)(\sqrt{x}+4)$. $x=4$.

(33) $x+\sqrt{2ax+x^2}=a$. $x=\frac{1}{4}a$.

(34) $\sqrt{4+\sqrt{x^4-x^2}} = x-2.$ $\qquad x = 2\frac{1}{8}.$

(35) $\sqrt[3]{ax+b} = \sqrt[3]{cx+d}.$ $\qquad x = \dfrac{d-b}{a-c}.$

(36) $\sqrt[n]{a+x} = \sqrt[2n]{x^2+5ax+b^2}.$ $\qquad x = \dfrac{a^2-b^2}{3a}.$

(37) $ax+b^2 = a^2+bx.$ $\qquad x = a+b.$

(38) $bx+2x-a = 3x+2c.$ $\qquad x = \dfrac{a+2c}{b-1}.$

(39) $\frac{5}{6}(x-a) - \frac{1}{5}(2x-3b) = 10a+11b.$ $\qquad x = 25a+24b.$

(40) $\dfrac{3x-a}{b} + \dfrac{x+2b}{c} = \dfrac{7x}{c} - \dfrac{a}{4}.$ $\qquad x = \dfrac{8b^2-4ac+abc}{12(2b-c)}.$

(41) $\dfrac{bx}{a} - \dfrac{d}{c} = \dfrac{a}{b} - \dfrac{cx}{d}.$ $\qquad x = \dfrac{ad}{bc}.$

(42) $(a+x)(b+x) - a(b+c) = \dfrac{a^2c}{b} + x^2.$ $\qquad x = \dfrac{ac}{b}.$

(43) $ax^2+a^3 = (ax+b^2)(a+x).$ $\qquad x = \dfrac{a(a^2-b^2)}{a^2+b^2}.$

(44) $(a+x)^{\frac{1}{2}} - \left(\dfrac{a}{a+x}\right)^{\frac{1}{2}} = (2a+x)^{\frac{1}{2}}.$

$$x = a^{\frac{1}{2}}\left(\frac{1-2a^{\frac{1}{2}}-a}{2+a^{\frac{1}{2}}}\right).$$

(45) $\dfrac{(a+x)^{\frac{1}{2}}+(a-x)^{\frac{1}{2}}}{(a+x)^{\frac{1}{2}}-(a-x)^{\frac{1}{2}}} = b^{\frac{1}{2}}.$ $\qquad x = \dfrac{2ab^{\frac{1}{2}}}{1+b^2}.$

(46) $\dfrac{(a+x^{\frac{1}{2}})^{\frac{1}{2}}}{x^{\frac{1}{4}}} + \dfrac{(a-x^{\frac{1}{2}})^{\frac{1}{2}}}{x^{\frac{1}{4}}} = x^{\frac{1}{4}}.$ $\qquad x = 4(a-1).$

(47) $\left(\dfrac{a^2}{x}+b\right)^{\frac{1}{2}} - \left(\dfrac{a^2}{x}-b\right)^{\frac{1}{2}} = c^{\frac{1}{2}}.$ $\qquad x = \dfrac{4a^2c}{4b^2+c^2}.$

(48) $\sqrt{a+x} - \sqrt{b} = \sqrt{x}.$ $\qquad x = \dfrac{a-b)^2}{4b}.$

(49) $\sqrt{x} + \sqrt{a + x} = \dfrac{na}{\sqrt{a + x}}$. $\qquad x = \dfrac{(n-1)^2}{2n-1}\,a.$

(50) $\sqrt{x + a} + \sqrt{x - a} = \dfrac{b}{\sqrt{x + a}}$. $\qquad x = \dfrac{2a^2 - 2ab + b^2}{2(b - a)}.$

(51) $\dfrac{1}{x^{\frac{1}{2}}} + \dfrac{1}{a^{\frac{1}{2}}} = \left\{\dfrac{1}{a} + \sqrt{\dfrac{4}{ax} + \dfrac{9}{x^2}}\right\}^{\frac{1}{2}}$. $\qquad x = 4a.$

(52) $\dfrac{5x - 9}{3 + \sqrt{5x}} - 1 = \frac{1}{2}(\sqrt{5x} - 3).$ $\qquad x = 5.$

(53) $\sqrt{x} + \sqrt{\{a - \sqrt{ax + x^2}\}} = \sqrt{a}.$ $\qquad x = \frac{9}{16}a.$

(54) $(x^{\frac{1}{2}} + 3a^2)^{\frac{1}{2}} - (x^{\frac{1}{2}} - 3a^2)^{\frac{1}{2}} = \dfrac{2a^{\frac{1}{2}}x^{\frac{1}{4}}}{b^{\frac{1}{2}}}$. $\qquad x = \dfrac{9a^3b^2}{4(b-a)}.$

PROBLEMS IN SIMPLE EQUATIONS.

1. What number is that to which if its third and fourth parts be added, the sum will exceed its sixth part by 17?

Answer: 12.

2. What number is that from which if 50 be subtracted, the remainder will be equal to its half, together with its fourth and sixth parts?

Answer: 600.

3. Find a number which when multiplied by 4 becomes as much above 30 as it is now below it.

Answer: 12.

4. Two persons at a distance of 240 leagues, set out to meet each other, and travel at the rates of 7 and 8 leagues a-day respectively: when and where will they meet?

Answer: 16 days; having travelled 112 and 128 leagues.

5. A labourer was engaged for 36 days upon the condition that for every day he worked he was to receive 2*s*. 6*d*.:

and for every day he was absent to forfeit 1*s.* 6*d.*: and at the end of his time he received £2. 18*s.* How many days did he work?

Answer: 28.

6. *A* has three times as much money as *B*, and if *B* give him £50., *A* will have four times as much as *B*: find the money of each.

Answer: *A*'s = £750, and *B*'s = £250.

7. *A* possesses £600. and *B* has £480: what sum must *A* receive of *B* that he may possess twice as much as *B*?

Answer: £120.

8. *A*'s money exceeds *B*'s and *C*'s by £240. and £320. respectively: and that of *B* and *C* together is £600: required the sum possessed by each.

Answer: *A*'s = £580, *B*'s = £340, *C*'s = £260.

9. *A*, *B*, *C* together possess £600: *A*, *B*, *D* together £720: *A*, *C*, *D* together £900: and *B*, *C*, *D* together have £1020: what is the sum of each?

Answer: *A*'s = £60, *B*'s = £180, *C*'s = £360, *D*'s = £480.

10. *A* and *B* together possess £150, and *C* has £50. more than *D*: also, *A* has twice as much as *C*, and *B* thrice as much as *D*: required the money of each.

Answer: *A*'s = £120, *B*'s = £30, *C*'s = £60, *D*'s = £10.

11. A merchant, after allowing £1600. for his annual expenditure, increases his property every year by a fourth part, and at the end of two years is £9000. richer than at first: what property does he begin with?

Answer: £22400.

12. From a sum of money is first taken away £20. more than its half: from the remainder £30. more than its third part: and from what then remained £40. more than its fourth part, and afterwards nothing remains: what is the sum?

Answer: £290.

13. *A* sold a certain number of tickets at a guinea each, and gave $\frac{1}{3}$ of the produce to *B*: $\frac{1}{4}$ of the remainder to *C*, and $\frac{1}{6}$ of the last remainder to *D*, after which he had £210. remaining: how many did he sell?

Answer: 500.

14. *A* has *m* times as much money as *B*: also, if they receive a£. and b£. respectively, *A* will have *n* times as much as *B*: what sum has each?

$$\text{Answer: } A\text{'s} = \frac{m(nb-a)}{m-n}\text{ £},\quad B\text{'s} = \frac{nb-a}{m-n}\text{ £}.$$

15. Given the sum of two quantities $= a$, and the sum of *m* times the former and *n* times the latter $= b$: to find them.

$$\text{Answer: } \frac{b-na}{m-n}, \text{ and } \frac{ma-b}{m-n}.$$

16. Divide a given quantity *a* into two parts, so that the sum of their quotients by *m* and *n* may $= b$.

$$\text{Answer: } \frac{m(a-nb)}{m-n}, \text{ and } \frac{n(mb-a)}{m-n}.$$

17. Divide a given magnitude *a* into three parts, so that the second may be *m* times and the third *n* times, as great as the first.

$$\text{Answer: } \frac{a}{1+m+n}, \quad \frac{ma}{1+m+n}, \text{ and } \frac{na}{1+m+n}.$$

18. *A* and *B* are possessed of certain sums of money, such that if they gain a£. and b£. respectively, *A* will be *m* times as rich as *B*: but if they gain c£. and d£. respectively, *A* becomes possessed of *n* times as much as *B*: required the money of each.

Answer:

$$A\text{'s} = \frac{m(nd-c)-n(mb-a)}{m-n}:\quad B\text{'s} = \frac{(nd-c)-(mb-a)}{m-n}.$$

19. Find the time in which three persons can jointly perform a piece of work, when they can separately do it in m, n and p days.

Answer : $\dfrac{mnp}{mn+mp+np}$ days.

20. A and B can do a piece of work in m days, A and C in n days, and B and C in p days: in what times can they accomplish it individually and collectively?

Answer :

A in $\dfrac{2mnp}{mp+np-mn}$ days; B in $\dfrac{2mnp}{mn+np-mp}$ days;

C in $\dfrac{2mnp}{mn+mp-np}$ days: A, B, C in $\dfrac{2mnp}{mn+mp+np}$ days.

PURE QUADRATIC EQUATIONS.

Solve the following pure quadratic equations.

(1) $11x^2 - 44 = 5x^2 + 10.$ $x = 3$, and $x = -3$.

(2) $7x^2 - 25 = 4x^2 - 13.$ $x = 2$, and $x = -2$.

(3) $(x + 2)^2 = 4x + 5.$ $x = 1$, and $x = -1$.

(4) $\frac{1}{3}(x^2 - 12) = \frac{1}{4}x^2 - 1.$ $x = 6$, and $x = -6$.

(5) $x\sqrt{6 + x^2} = 1 + x^2.$ $x = \frac{1}{2}$, and $x = -\frac{1}{2}$.

(6) $\sqrt{x - a} = \sqrt{x + \sqrt{b^2 + x^2}}.$ $x = \pm\sqrt{a^2 - b^2}.$

(7) $\dfrac{a}{x} + \dfrac{\sqrt{a^2 - x^2}}{x} = \dfrac{x}{b}.$ $x = \pm\sqrt{2ab - b^2}.$

(8) $\dfrac{1}{a - \sqrt{a^2 - x^2}} - \dfrac{1}{a + \sqrt{a^2 - x^2}} = \dfrac{a}{x^2}.$ $x = \pm\frac{1}{2}a\sqrt{3}.$

(9) $\dfrac{2}{x + \sqrt{2 - x^2}} + \dfrac{2}{x - \sqrt{2 - x^2}} = x.$ $x = \pm\sqrt{3}.$

(10) $\dfrac{\sqrt{a^2 + x^2} + x}{\sqrt{a^2 + x^2} - x} = \dfrac{b}{c}.$ $x = \pm\dfrac{a(b - c)}{2\sqrt{bc}}.$

(11) $\dfrac{a+x+\sqrt{2ax+x^2}}{a+x-\sqrt{2ax+x^2}}=b.$ $\qquad x=\pm\dfrac{a(\sqrt{b}\pm 1)^2}{2\sqrt{b}}.$

(12) $\dfrac{ax+1+\sqrt{a^2x^2-1}}{ax+1-\sqrt{a^2x^2-1}}=\frac{1}{2}b^2x.$ $\qquad x=\pm\dfrac{2}{b\sqrt{4a-b^2}}.$

(13) $\sqrt{(b+c)^2+x^2}+\sqrt{(b-c)^2+x^2}=2a.$

$$x=\pm\frac{\sqrt{(a^2-b^2)(a^2-c^2)}}{a}.$$

(14) $\dfrac{\sqrt{a+bx^n}+\sqrt{a-bx^n}}{\sqrt{a+bx^n}-\sqrt{a-bx^n}}=c.$ $\qquad x=\left\{\dfrac{2ac}{b(1+c^2)}\right\}^{\frac{1}{n}}.$

(15) $\dfrac{(a+x)^{\frac{1}{m}}}{a}+\dfrac{(a+x)^{\frac{1}{m}}}{x}=\dfrac{x^{\frac{1}{m}}}{c}.$ $\qquad x=\dfrac{ac^{\frac{m}{m+1}}}{a^{\frac{m}{m+1}}-c^{\frac{m}{m+1}}}.$

ADFECTED QUADRATIC EQUATIONS.

Find the roots of the following equations.

(1) $x^2-10x=24.$ $\qquad x=12$, and -2.

(2) $x^2-12x+20=0.$ $\qquad x=10$, and 2.

(3) $x^2-14=13x.$ $\qquad x=14$, and -1.

(4) $5x^2-24x=5.$ $\qquad x=5$, and $-\frac{1}{5}$.

(5) $9x-5x^2=2\frac{1}{4}.$ $\qquad x=1\frac{1}{2}$, and $\frac{3}{10}$.

✓ (6) $\frac{1}{3}x^2-\frac{1}{2}x=9.$ $\qquad x=6$, and $-4\frac{1}{2}$.

✓ (7) $2x=4+\dfrac{6}{x}.$ $\qquad x=3$, and -1.

(8) $\dfrac{4x}{5-x}-\dfrac{20-4x}{x}=15.$ $\qquad x=4$, and $-1\frac{2}{3}$.

✓ (9) $x-\dfrac{x^3-8}{x^2+5}=2.$ $\qquad x=2$, and $\frac{1}{2}$.

(10) $\dfrac{x}{x+1}+\dfrac{x+1}{x}=2\frac{1}{6}.$ $\qquad x=2$, and -3.

(11) $\dfrac{x+4}{x-3} - \dfrac{2x-3}{x+4} = 7\tfrac{3}{8}$. $x = 4$, and $-2\tfrac{57}{67}$.

(12) $\dfrac{x+7}{x+11} - \dfrac{x+5}{x+12} = \dfrac{47}{306}$. $x = 6$, and $-9\tfrac{22}{47}$.

(13) $\dfrac{x-1}{x+1} + \dfrac{x+3}{x-3} = 2\left(\dfrac{x+2}{x-2}\right)$. $x = 5$, and 0.

(14) $\dfrac{5x+36}{10x-81} + \dfrac{3}{25} = \dfrac{8x}{5x-8}$. $x = 14\tfrac{3}{5}$, and $\tfrac{24}{83}$.

(15) $\dfrac{3x}{x+2} - \dfrac{x-1}{6} = x - 9$. $x = 10$, and $-1\tfrac{4}{7}$.

(16) $x + \dfrac{24}{x-1} = 3x - 4$. $x = 5$, and -2.

(17) $(2x+3)^{\frac{1}{2}} \times (3x+7)^{\frac{1}{2}} = 12$. $x = 3$, and $-6\tfrac{5}{6}$.

(18) $2x^{\frac{1}{2}} + 2x^{-\frac{1}{2}} = 5$. $x = 4$, and $\tfrac{1}{4}$.

(19) $x - 4 = (x^{\frac{1}{2}} + 2)(x - 8)$. $x = 9$, and 4.

(20) $3\sqrt{112-8x} = 19 + \sqrt{3x+7}$. $x = 6$, and $11\tfrac{523}{625}$.

(21) $(x^2 + \sqrt{x^4 + 22x^2 + 32x})^{\frac{1}{2}} = x + 2$. $x = \pm 2^{\frac{1}{2}}$, and $\pm(-8)^{\frac{1}{2}}$.

(22) $(9 + 5\sqrt{3})\,x^2 - (15 + 7\sqrt{3})\,x + 6 = 0$.

$x = 2 - \sqrt{3}$, and $3 - \sqrt{3}$.

(23) $x^2 - (a+b)\,x + ab = 0$. $x = a$, and b.

(24) $ax^2 - 2ab^{\frac{1}{2}}x = bx^2 - ab$. $x = \dfrac{(ab)^{\frac{1}{2}}}{a^{\frac{1}{2}} \mp b^{\frac{1}{2}}}$.

(25) $(a-b)\,x^2 - (a+b)\,x + 2b = 0$. $x = 1$, and $\dfrac{2b}{a-b}$.

(26) $(a^2 - b^2)\,x - (a+b)^2\,(a^{\frac{1}{2}} - b^{\frac{1}{2}}) = (ab^{\frac{1}{2}} - a^{\frac{1}{2}}b)\,x^2$.

$x = \dfrac{a+b}{a^{\frac{1}{2}}}$, and $\dfrac{a+b}{b^{\frac{1}{2}}}$.

(27) $ab\,(c^2 + x^2) = (a^2 + b^2)\,cx$. $x = \dfrac{ac}{b}$, and $\dfrac{bc}{a}$.

(28) $a^2(1+b^2x^2)=b(2a^2x+b)$. $\quad x=\dfrac{a+b}{ab}$, and $\dfrac{a-b}{ab}$.

(29) $(x-c)\sqrt{ab}-(a-b)\sqrt{cx}=0$. $\quad x=\dfrac{ac}{a}$, and $\dfrac{bc}{a}$.

(30) $a^3x^2+(1+b)ab^{\frac{1}{2}}+a^2bx^2=\{a^3b^{\frac{1}{2}}+(a+b)(1+b)\}x$.

$$x=\frac{ab^{\frac{1}{2}}}{a+b},\ \text{and}\ \frac{1+b}{a^2}.$$

(31) $1+\{(2x+7)(4\sqrt{x}-7)\}^{\frac{1}{3}}=2\sqrt{x}$. $\quad x=9$, and 64.

(32) $x+4+\left(\dfrac{x+4}{x-4}\right)^{\frac{1}{2}}=\dfrac{12}{x-4}$. $\quad x=\pm 5$, and $\pm 4\sqrt{2}$.

(33) $2\sqrt{x}+\sqrt{4x+\sqrt{7x+2}}=1$. $\quad x=1$, and $\frac{1}{81}$.

(34) $\dfrac{5(3x-1)}{1+5\sqrt{x}}+\dfrac{2}{\sqrt{x}}=3\sqrt{x}$. $\quad x=\frac{1}{9}$, and 4.

(35) $\sqrt{a+x}+\sqrt{a-x}=\dfrac{12a}{5\sqrt{a+x}}$. $\quad x=\dfrac{4a}{5}$, and $\dfrac{3a}{5}$.

(36) $(3+x^{\frac{1}{2}})^{\frac{1}{2}}+(4-x^{\frac{1}{2}})^{\frac{1}{2}}=(7+2x^{\frac{1}{2}})^{\frac{1}{2}}$. $\quad x=\dfrac{49\pm\sqrt{97}}{8}$.

(37) $(a^{4m}+1)(x^{\frac{1}{2}}-1)^2=2(x+1)$. $\quad x=\left(\dfrac{a^{2m}\pm 1}{a^{2m}\mp 1}\right)^2$.

(38) $x^{\frac{p+q}{2pq}}-\frac{1}{2}\left(\dfrac{a^2-b^2}{a^2+b^2}\right)\left(x^{\frac{1}{p}}+x^{\frac{1}{q}}\right)$. $\quad x=0$, and $\left(\dfrac{a\pm b}{a\mp b}\right)^{\frac{2pq}{q-p}}$.

(39) $x^2+\dfrac{1}{x^2}+x+\dfrac{1}{x}=4$. $\quad x=1$, and $\frac{1}{2}(3\pm\sqrt{5})$.

(40) $x^2-2x^{\frac{3}{2}}+2x-x^{\frac{1}{2}}=6$. $\quad x=1,\ 4$, and $\frac{1}{2}(\pm\sqrt{-11}-5)$.

(41) $a^2b^2x^{\frac{1}{n}}-4(ab)^{\frac{3}{2}}x^{\frac{m+n}{2mn}}=(a-b)^2x^{\frac{1}{m}}$.

$$x=\left(\frac{1}{\sqrt{a}}\pm\frac{1}{\sqrt{b}}\right)^{\frac{4mn}{m-n}}.$$

(42) $x^2(x+4)+2x(x+4)=2-(x+4)$. $x=-2$, and $-2\pm\sqrt{3}$.

(43) $\dfrac{\sqrt{1+x}}{1+\sqrt{1+x}}=\dfrac{\sqrt{1-x}}{1-\sqrt{1-x}}$. $x=0$, and $\pm\frac{1}{2}\sqrt{3}$.

(44) $x^2-2x+6(x^2-2x+5)^{\frac{1}{2}}=11$. $x=1$, and $1\pm2\sqrt{15}$.

(45) $\dfrac{x-18}{x^{\frac{1}{2}}-18^{\frac{1}{2}}}+\dfrac{(x-18)^{\frac{1}{2}}}{x^{\frac{1}{2}}}=x^{\frac{1}{2}}-\dfrac{x^{\frac{1}{2}}}{(x-18)^{\frac{1}{2}}}$. $x=9\pm\dfrac{27}{7^{\frac{1}{2}}}$.

(46) $2x^2-2x+2\sqrt{2x^2-7x+6}=5x-6$.

$x=2$, $1\frac{1}{2}$, and $\frac{1}{4}(7\pm\sqrt{33})$.

(47) $x^2-x+5\sqrt{2x^2-5x+6}=\frac{1}{2}(3x+33)$.

$x=3$, $-\frac{1}{2}$, and $\frac{1}{4}(5\pm\sqrt{1329})$.

(48) $ax=(\sqrt{1+x}-1)(\sqrt{1-x}+1)$.

$x=0$, and $\dfrac{4a(1-a^2)}{(1+a^2)^2}$.

(49) $x^2(x-2)^2+5x^2(x-2)=2(x-2)$.

$x=2$, -1, and $\pm\sqrt{3}-1$.

(50) $(x-4)^2+2(x-4)=\dfrac{2}{x}-1$. $x=2$, and $2\pm\sqrt{3}$.

(51) $x^2(x^2-18)=4(x-12)$. $x=\pm4$, and ±2.

(52) $2x^3-x^2=1$. $x=1$, and $\frac{1}{4}(1\pm\sqrt{-7})$.

(53) $(x+1)\sqrt{x}=2$. $x=1$, and $\frac{1}{2}(-3\pm\sqrt{-7})$.

(54) $(x-3)x=3+4\sqrt{x}$.

$x=\frac{1}{2}(7\pm\sqrt{13})$, and $\frac{1}{2}(-1\mp\sqrt{-3})$.

(55) $x^2-\dfrac{2}{3x}=1\frac{4}{9}$. $x=-\frac{2}{3}$, and $\frac{1}{3}(1\pm\sqrt{10})$.

(56) $\frac{1}{2}x(x+1)=24+7x^{\frac{1}{2}}$.

$x=9$, 4, and $\frac{1}{2}(-15\pm\sqrt{-31})$.

PROBLEMS IN QUADRATIC EQUATIONS.

1. Find two numbers, one of which is $\frac{3}{5}$ths of the other, so that the difference of their squares may be 16^2.

Answer: ± 20, and ± 12.

2. Required two numbers whose difference is 8, and product 128.

Answer: ± 8, and ± 16.

3. Determine two magnitudes whose difference is $\frac{1}{6}$, and the sum of whose squares is $\left(\frac{5}{6}\right)^2$.

Answer: $\pm \frac{1}{2}$, and $\pm \frac{2}{3}$.

4. Find three magnitudes, the products of each two of which are p, q and r, respectively.

Answer: $\pm \left(\frac{pq}{r}\right)^{\frac{1}{2}}$, $\pm \left(\frac{pr}{q}\right)^{\frac{1}{2}}$, and $\pm \left(\frac{qr}{p}\right)^{\frac{1}{2}}$.

5. The reckoning of a party at a tavern was £3. 12*s*., but in consequence of two of them having no money, each of the rest paid 6*d*. more than he otherwise should have done: required their number. Answer: 18.

6. Divide the given quantity a into two parts, so that the sum of their square roots may be $b^{\frac{1}{2}}$.

Answer: $\frac{1}{2}(a \pm \sqrt{2ab - b^2})$, and $\frac{1}{2}(a \mp \sqrt{2ab - b^2})$.

7. Given a, the product of two magnitudes, and b, the difference of the products by c and d: to find them.

Answer: $\frac{1}{2c}(\pm \sqrt{b^2 + 4acd} + b)$, and $\frac{1}{2d}(\pm \sqrt{b^2 + 4acd} - b)$.

8. It is required to divide each of the numbers 11 and 17 into two parts, so that the product of the first parts of each may be 45, and of the second 48.

Answer: 5, 6, and 9, 8.

9. Divide each of the numbers 21 and 30 into two parts, so that the first part of 21 may be three times as great as the first part of 30: and that the sum of the squares of the remaining parts may be 585.

Answer: 18, 3, and 6, 24.

10. To divide each of the numbers 19 and 29 into two parts, so that the difference of the squares of the first parts of each may be 72, and the difference of the squares of the remaining parts 180.

Answer: 7, 12, and 11, 18.

11. It is required to divide each of the numbers 17, 23, and 38 into two parts, so that the product of one part of 17 and one part of 23 may be 63: the product of the other part of 17 and one part of 38 may be 180, and the product of the remaining parts of 23 and 38 may be 280.

Answer: 7, 10: 9, 14, and 18, 20.

12. A grazier bought a certain number of oxen for £240, and after losing 3, sold the remainder for £8. a head more than they cost him, thus gaining £59. by his bargain. What number did he buy?

Answer: 16.

13. *A* and *B* engaged to reap equal quantities of wheat, and *A* began half an hour before *B*: at 12 o'clock they rested an hour, having finished half the work: also, *B*'s part was finished at 7 o'clock, and *A*'s at a quarter before 10: determine the times at which they commenced.

Answer: $4\frac{1}{2}$ and 5.

14. *A* and *B* start from two places *C* and *D* at the same time, *A* from *C* intending to pass through *D*, and *B* from *D* travelling the same way: when *A* overtakes *B*, it is found that they had together travelled 30 miles: that *A* had passed through *D* four hours before, and that *B* was 9 hours journey from *C*: find the distance between *C* and *D*, and the rates of travelling of *A* and *B*.

Answer: 6 miles: and 3 and 2 miles per hour.

SIMULTANEOUS EQUATIONS.

It is required to solve the following equations.

1. $x + y = 9$ and $3x + 5y = 35$. $\quad x = 5,\ y = 4$.

2. $2x + 3y = 18$ and $3x - 2y = 1$. $\quad x = 3,\ y = 4$.

3. $2x - 9y = 11$ and $3x - 12y = 15$. $\quad x = 1,\ y = -1$.

4. $3x - 7y = 7$ and $11x + 5y = 87$. $\quad x = 7,\ y = 2$.

5. $9x - 4y = 8$ and $13x + 7y = 101$. $\quad x = 4,\ y = 7$.

6. $x - \frac{1}{7}(y - 2) = 5$ and $4y - \frac{1}{3}(x + 10) = 3$. $\quad x = 5,\ y = 2$.

7. $\dfrac{4x + 3y}{6} = 8$ and $\dfrac{7y - 3x}{2} - y = 11$. $\quad x = 6,\ y = 8$.

8. $2x - \dfrac{y - 3}{5} = 4$ and $3y + \dfrac{x - 2}{3} = 9$. $\quad x = 2,\ y = 3$.

9. $\dfrac{4x + 5y}{40} = x - y$ and $\dfrac{2x - y}{3} + 2y = \frac{1}{2}$. $\quad x = \frac{1}{4},\ y = \frac{1}{5}$.

10. $$\left.\begin{aligned} &\tfrac{1}{7}(2x - y) + 3x = 2y - 6, \\ &\tfrac{1}{5}\ (y + 3) + \tfrac{1}{6}(y - x) = 2x - 8. \end{aligned}\right\} \quad x = 6,\ y = 12.$$

11. $$\left.\begin{aligned} &\tfrac{1}{11}(4x + 2y) = 6 - \tfrac{1}{4}(5y - 3x), \\ &\tfrac{1}{3}(8y - 10) = \tfrac{1}{6}(5x + 3y) + 5. \end{aligned}\right\} \quad x = 3,\ y = 5.$$

12. $$\left.\begin{aligned} &\tfrac{1}{13}(5x - 6y) + 3x = 4y - 2, \\ &\tfrac{1}{6}(5x + 6y) - \tfrac{1}{4}(3x - 2y) = 2y - 2. \end{aligned}\right\} \quad x = 6,\ y = 5.$$

13. $$\left.\begin{aligned} &(x + 5)(y + 7) = (x + 1)(y - 9) + 112, \\ &2x + 10 = 3y + 1. \end{aligned}\right\} \quad x = 3,\ y = 5.$$

14. $$\left.\begin{aligned} &\frac{3x - 2y}{3} + \frac{11y - 2}{8} = \frac{4x - 3y + 5}{7} + \frac{45 - x}{5}, \\ &45 - \frac{4x - 2}{3} = \frac{55x + 71y + 1}{18}. \end{aligned}\right\} \quad x = 5,\ y = 6.$$

15. $$\left.\begin{aligned} &y^{\frac{1}{2}} - (a - x)^{\frac{1}{2}} = (y - x)^{\frac{1}{2}}, \\ &2(y - x)^{\frac{1}{2}} = 3(a - x)^{\frac{1}{2}}. \end{aligned}\right\} \quad x = \tfrac{4}{5}a,\ y = \tfrac{5}{4}a.$$

16. $$\left.\begin{aligned} &\frac{4x - 8y + 1}{2} = \frac{10x^2 - 12y^2 - 14xy + 2x}{5x + 3y + 3}, \\ &2\sqrt{6 + x} = 3\sqrt{6 - y}. \end{aligned}\right\} \quad x = 3,\ y = 2.$$

17. $\left.\begin{aligned} 3x + \tfrac{2}{3}(xy^2 + 9x^2y)^{\frac{1}{2}} &= (x - \tfrac{1}{3})\,y, \\ 18x - 2y &= xy. \end{aligned}\right\}$ $x = 4, \ y = 12.$

18. $x^2 + xy + y^2 = 52$ and $xy - x^2 = 8.$ $x = \pm 2, \ y = \pm 6.$

19. $x^2 - 2xy - y^2 = 31$ and $\frac{1}{2}x^2 + 2xy - y^2 = 101.$

$x = \pm 10, \ y = \pm 3.$

20. $\left.\begin{aligned} y^2 + y + 17x &= 54, \\ \sqrt{x^2 + 2y^2} + x &= 8. \end{aligned}\right\}$ $x = 2$ and $2\frac{64}{81}$: $y = \pm 4$ and $\pm\frac{28}{81}.$

21. $x^4 + y^4 = 97$ and $x + y = 5.$ $x = 2, \ y = 3.$

22. $x^{\frac{3}{2}} + y^{\frac{2}{3}} = 3x$ and $x^{\frac{1}{2}} + y^{\frac{1}{3}} = x.$ $x = 4$ and 1, $y = 8.$

23. $\left.\begin{aligned} (5x^{\frac{1}{2}} + 5y^{\frac{1}{2}})^{\frac{1}{2}} + y^{\frac{1}{2}} &= 10 - x^{\frac{1}{2}}, \\ x^{\frac{5}{2}} + y^{\frac{5}{2}} &= 275. \end{aligned}\right\}$ $x = 9$ and 4: $y = 4$ and $9.$

24. $y - y^{\frac{1}{2}} = 16 - x$ and $28 - y = x + 4x^{\frac{1}{2}}.$ $x = 4, \ y = 16.$

25. $\left.\begin{aligned} 2x + y &= 26 - 7(2x + y + 4)^{\frac{1}{2}}, \\ \frac{2x + y^{\frac{1}{2}}}{2x - y^{\frac{1}{2}}} &= \frac{16}{15} + \frac{2x - y^{\frac{1}{2}}}{2x + y^{\frac{1}{2}}}. \end{aligned}\right\}$ $x = 2$ and -10 $y = 1$ and $25.$

26. $\left.\begin{aligned} x + y + \sqrt{xy} &= 28, \\ x^2 + y^2 + xy &= 336. \end{aligned}\right\}$ $x = \pm 16, \ y = \pm 4.$

27. $\left.\begin{aligned} \frac{x}{x + y} - \frac{y}{x} &= \frac{x^2 - y^2}{18}, \\ \frac{x}{y} - \frac{x + y}{x} &= \frac{y}{x}. \end{aligned}\right\}$ $x = \pm 2, \ y = \pm 1.$

28. $\left.\begin{aligned} x^4 - x^2 + y^4 - y^2 &= 84, \\ x^2 + x^2y^2 + y^2 &= 49. \end{aligned}\right\}$ $x = \pm 3, \ y = \pm 2.$

29. $\left.\begin{aligned} y^{\frac{1}{2}} - (y - x)^{\frac{1}{2}} &= (20 - x)^{\frac{1}{2}}, \\ 2(y - x)^{\frac{1}{2}} &= 3(20 - x)^{\frac{1}{2}}. \end{aligned}\right\}$ $x = 16$ and 40: $y = 25$ and $-5.$

30. $\left.\begin{aligned} \frac{y}{x} - \frac{81}{xy} &= (2y + 18)\frac{x^{\frac{1}{2}}}{y}, \\ y + 3x^{\frac{3}{2}} &= 9 + x^{\frac{3}{2}}y^{\frac{1}{2}}. \end{aligned}\right\}$ $x = 4, \ y = 25.$

31. $\left.\begin{aligned} &\left(\frac{x}{y}\right)^{\frac{1}{2}} + \left(\frac{y}{x}\right)^{\frac{1}{2}} = \frac{61}{(xy)^{\frac{1}{2}}} + 1, \\ &\sqrt[4]{x^3y} + \sqrt[4]{xy^3} = 78. \end{aligned}\right\}$ $x = 81$ and 16: $y = 16$ and 81.

32. $\left.\begin{aligned} &\frac{x+\sqrt{x^2-y^2}}{x-\sqrt{x^2-y^2}} = 4\tfrac{1}{4} - \frac{x-\sqrt{x^2-y^2}}{x+\sqrt{x^2-y^2}}, \\ &x(x+y) = 52 - \sqrt{x^2+xy+4}. \end{aligned}\right\}$ $x = \pm 5$, $y = \pm 4$.

33. $\left.\begin{aligned} &\frac{x+y+\sqrt{x^2-y^2}}{x+y-\sqrt{x^2-y^2}} = \frac{9(x+y)}{8y}, \\ &(x^2+y)^2+x-y=2x(x^2+y)+506. \end{aligned}\right\}$ $x=5$ and $-4\tfrac{3}{5}$: $y=3$ and $-2\tfrac{19}{25}$.

34. $\left.\begin{aligned} &5y+\tfrac{1}{5}\sqrt{x^2-15y-14}=\tfrac{1}{3}x^2-36, \\ &\frac{x^2}{8y}+\frac{2x}{3}=\sqrt{\frac{x^3}{3y}+\frac{x^2}{4}-\frac{y}{2}}. \end{aligned}\right\}$ $x = 12$ and $-9\tfrac{1}{3}$: $y = 2$ and $-1\tfrac{7}{12}$.

35. $\left.\begin{aligned} &x^2y - 4 = 4x^{\frac{1}{2}}y - \tfrac{1}{4}y^3, \\ &x^{\frac{3}{2}} - 3 = x^{\frac{1}{2}}y^{\frac{1}{2}}(x^{\frac{1}{2}} - y^{\frac{1}{2}}). \end{aligned}\right\}$ $x = 1$, $y = 4$.

36. $2a(y-3b)=b(2a-x)$ and $ay=bx$. $x=\frac{8}{3}a$, $y=\frac{8}{3}b$.

37. $\left.\begin{aligned} &(a^2-b^2)(3x+5y)=(4a-b)2ab, \\ &a^2x-\frac{ab^2c}{a+b}+(a+b+c)by=b^2x+(a+2b)ab. \end{aligned}\right\}$ $x=\dfrac{ab}{a-b}$: $y=\dfrac{ab}{a+b}$.

38. $\left.\begin{aligned} &x(bc-xy)=y(xy-ac), \\ &xy(ay+bx-xy)=abc(x+y-c). \end{aligned}\right\}$ $x=\pm\sqrt{ac}$: $y=\pm\sqrt{bc}$.

39. $\frac{x}{a}+\frac{y}{b}=1-\frac{x}{c}$ and $\frac{y}{a}+\frac{x}{b}=1+\frac{y}{c}$.

$$x=\frac{abc(ac+ab-bc)}{a^2b^2+a^2c^2-b^2c^2},\quad y=\frac{abc(ac-ab-bc)}{a^2b^2+a^2c^2-b^2c^2}.$$

40. $5x + 3y = 65,\ \ 2y - z = 11,\ \ 3x + 4z = 57.$

$$x = 7,\ \ y = 10,\ \ z = 9.$$

41. $x + 50 = y + z,\ \ y + 50 = 2x + 3z,\ \ z + 50 = 3x + 4y.$

$$x = -3\tfrac{1}{33},\ \ y = 21\tfrac{7}{33},\ \ z = 25\tfrac{25}{33}.$$

42. $3x + 2y - z = 20,\ \ 2x + 3y + 6z = 70,\ \ x - y + 6z = 41.$

$$x = 5,\ \ y = 6,\ \ z = 7.$$

43. $\frac{1}{x} + \frac{1}{y} = a,\ \ \frac{1}{x} + \frac{1}{z} = b,\ \ \frac{1}{y} + \frac{1}{z} = c.$

$$x = \frac{2}{a+b-c},\ \ y = \frac{2}{a+c-b},\ \ z = \frac{2}{b+c-a}.$$

44. $\frac{3}{x} - \frac{4}{5y} + \frac{1}{z} = 7\tfrac{3}{5},\ \ \frac{1}{3x} + \frac{1}{2y} + \frac{2}{z} = 10\tfrac{1}{6},\ \ \frac{4}{5x} - \frac{1}{2y} + \frac{4}{z} = 16\tfrac{1}{10}.$ $\qquad x = \tfrac{1}{2},\ \ y = \tfrac{1}{3},\ \ z = \tfrac{1}{4}.$

45. $x + y + z = 14,\ \ x^2 + y^2 + z^2 = 84,$ and $xz = y^2.$

$$x = 2,\ \ y = 4,\ \ z = 8.$$

46. $xy = a(x + y),\ \ xz = b(x + z),$ and $yz = c(y + z).$

$$x = \frac{2abc}{ac + bc - ab},\ \ y = \frac{2abc}{ab + bc - ac},\ \ z = \frac{2abc}{ab + ac - bc}.$$

47. $x(y + z) = a,\ \ y(x + z) = b,$ and $z(x + y) = c.$

$$x = \pm\sqrt{\frac{(a-c+b)(a-b+c)}{2(c-a+b)}}:$$

$$y = \pm\sqrt{\frac{(b-c+a)(b-a+c)}{2(a-b+c)}}:$$

$$z = \pm\sqrt{\frac{(c-a+b)(c-b+a)}{2(b-c+a)}}.$$

48. $xyz = a^2(x+y)$, $xyz = b^2(y+z)$, and $xyz = c^2(x+z)$.

$$x = \pm abc\sqrt{\frac{2(a^2b^2 + b^2c^2 - a^2c^2)}{(a^2b^2 + a^2c^2 - b^2c^2)(b^2c^2 + a^2c^2 - a^2b^2)}}:$$

$$y = \pm abc\sqrt{\frac{2(b^2c^2 + a^2c^2 - a^2b^2)}{(a^2b^2 + a^2c^2 - b^2c^2)(a^2b^2 + b^2c^2 - a^2c^2)}}:$$

$$z = \pm abc\sqrt{\frac{2(a^2b^2 + a^2c^2 - b^2c^2)}{(a^2b^2 + b^2c^2 - a^2c^2)(b^2c^2 + a^2c^2 - a^2b^2)}}.$$

49. $x(x+y+z) = a^2$, $y(x+y+z) = b^2$, $z(x+y+z) = c^2$.

$$x = \frac{\pm a^2}{(a^2+b^2+c^2)^{\frac{1}{2}}},\quad y = \frac{\pm b^2}{(a^2+b^2+c^2)^{\frac{1}{2}}},\quad z = \frac{\pm c^2}{(a^2+b^2+c^2)^{\frac{1}{2}}}.$$

50. $xy = a$, $xz = b$, $xu = c$, and $xyzu = d$.

$$x = \pm\left(\frac{abc}{d}\right)^{\frac{1}{2}},\quad y = \pm\left(\frac{ad}{bc}\right)^{\frac{1}{2}},\quad z = \pm\left(\frac{bd}{ac}\right)^{\frac{1}{2}},\quad u = \pm\left(\frac{cd}{ab}\right)^{\frac{1}{2}}.$$

51.
$$u + ax + a^2y + a^3z + a^4 = 0:$$
$$u + bx + b^2y + b^3z + b^4 = 0:$$
$$u + cx + c^2y + c^3z + c^4 = 0:$$
$$u + dx + d^2y + d^3z + d^4 = 0.$$

$$u = abcd,\quad x = -(abc + abd + acd + bcd),$$
$$y = ab + ac + ad + bc + bd + cd,\quad z = -(a + b + c + d).$$

52.
$$x_2 + x_3 + x_4 + \&c. + x_m = a_1:$$
$$x_1 + x_3 + x_4 + \&c. + x_m = a_2:$$
$$x_1 + x_2 + x_4 + \&c. + x_m = a_3:\ \&c.$$
$$x_1 + x_2 + x_3 + \&c. + x_{m-1} = a_m.$$

$$x_1 = \frac{1}{m-1}\{a_2 + a_3 + a_4 + \&c. + a_m - (m-2)a_1\}:$$

$$x_2 = \frac{1}{m-1}\{a_1 + a_3 + a_4 + \&c. + a_m - (m-2)a_2\}:$$

$$x_3 = \frac{1}{m-1}\{a_1 + a_2 + a_4 + \&c. + a_m - (m-2)a_3\}:\ \&c.$$

$$x_m = \frac{1}{m-1}\{a_1 + a_2 + a_3 + \&c. + a_{m-1} - (m-2)a_m\}.$$

PROBLEMS IN SIMULTANEOUS EQUATIONS.

1. The sum of two numbers is 17, and the product of one of them and its excess above the other is 55: find them.

Answer: 11 and 6.

2. Find a fraction such that if its numerator be increased and its denominator be diminished by 1, the result is $\frac{4}{5}$: but if its numerator be diminished and its denominator be increased by 1, the result is $\frac{1}{2}$.

Answer: $\frac{7}{11}$.

3. Required two numbers whose product is 48, and the difference of whose squares is 28.

Answer: 6 and 8.

4. Given the difference of two quantities $= a$, and the difference between the sum of their squares and their product $= b^2$, to find them.

Answer: $\frac{1}{2}(\sqrt{4b^2 - 3a^2} + a)$, and $\frac{1}{2}(\sqrt{4b^2 - 3a^2} - a)$.

5. What two magnitudes are those, whose sum, quotient and difference of their squares, are equal to each other?

Answer: $\frac{1}{2}(2 + \sqrt{2})$, and $\frac{1}{2}\sqrt{2}$.

6. Find two magnitudes so that their product, the difference of their squares, and the quotient of their cubes, may all be equal to one another.

Answer: $\frac{1}{2}(3 + \sqrt{5})$, and $\frac{1}{2}(1 + \sqrt{5})$.

7. There are two magnitudes whose product is equal to the difference of their squares, and the sum of their squares is also equal to the difference of their cubes; find them.

Answer: $\frac{1}{2}\sqrt{5}$, and $\frac{1}{4}(5 + \sqrt{5})$.

8. Required two numbers whose sum multiplied by the sum of their squares is 272, and whose difference multiplied by the difference of their squares is 32.

Answer: 5 and 3.

9. Divide each of the numbers 21 and 35 into two parts, so that the difference of the squares of the first parts may be 57, and that of the second parts 407.

Answer: 8, 13, and 11, 24.

10. Find two numbers such that the sum of their squares multiplied by the less and divided by the greater is $83\frac{1}{5}$: and the difference of the squares multiplied by the greater and divided by the less is 1920.

Answer: 20 and 4.

11. Find two numbers whose sum is 5, such that the product of the sums of their squares and cubes may be 455.

Answer: 3 and 2.

12. Required four magnitudes whose products, taken three and three together, are a^3, b^3, c^3, and d^3.

Answer: $\dfrac{abc}{d^2}$, $\dfrac{abd}{c^2}$, $\dfrac{acd}{b^2}$ and $\dfrac{bcd}{a^2}$.

13. There are four numbers such that if each be multiplied by their sum, the products are 252, 504, 396 and 144: find them.

Answer: 7, 14, 11 and 4.

14. Find four quantities, such that the first with half of the rest, the second with a third of the rest, the third with a fourth of the rest, and the fourth with a fifth of the rest, may each be equal to a.

Answers: $\dfrac{1}{37}a$, $\dfrac{19}{37}a$, $\dfrac{25}{37}a$ and $\dfrac{28}{37}a$.

15. The sum of four numbers is 52; the sum of the products of the first and second, and third and fourth is 360: of the first and third, and second and fourth 315: and of the first and fourth, and second and third 280: find them.

Answer: 21, 14, 11 and 6.

On these subjects, the Student is further referred to *Bland's* Algebraical Problems.

RATIO, PROPORTION AND VARIATION.

1. Shew that the ratio $a^2 - x^2 : a^2 + x^2$ is greater than the ratio $a - x : a + x$.

2. Prove that $x^3+y^3 : x^2+y^2$ is greater than $x^2+y^2 : x+y$.

3. Shew that the ratio $4a^3-3a^2x-4ax^2+3x^3 : 3a^3-2a^2x - 3ax^2 + 2x^3$, is equal to the ratio $4a-3x : 3a-2x$.

4. What quantity must be added to each of the terms of the ratio $a : b$, that it may become equal to $c : d$?

Answer: $\dfrac{ad - bc}{c - d}$.

5. What is the ratio resulting from the composition of $a^2 + b^2 : a^2 - b^2$ and $(a + b)^2 : (a - b)^2$?

Answer: $(a + b)(a^2 + b^2) : (a - b)^3$.

6. If x be to y in the duplicate ratio of a to b, and a be to b in the subduplicate ratio of $a + x : a - y$: then will $2x : a = x - y : y$.

7. If $a : b = c : d$, it is required to prove that

$$(a + d) - (b + c) = \frac{(a - b)(a - c)}{a}:$$

$$(a + b) - (c + d) = \frac{(a + b)(b - d)}{b}:$$

$$\left(\frac{1}{a} + \frac{1}{d}\right) - \left(\frac{1}{b} - \frac{1}{c}\right) = \frac{(a - b)(a - c)}{abc}.$$

8. The arithmetic mean between two numbers exceeds the geometric by 13, and the geometric exceeds the harmonic by 12: what are the numbers?

Answer: 104 and 234.

9. If the arithmetic mean between a and b be twice as great as the geometric mean: prove that

$$a : b = 2 + \sqrt{3} : 2 - \sqrt{3}.$$

10. If the arithmetic mean between a and b be m times the harmonic, then will $\dfrac{a}{b} = \dfrac{\sqrt{m} + \sqrt{m-1}}{\sqrt{m} - \sqrt{m-1}}$.

11. If the geometric mean between a and b be m times the harmonic, then will $\dfrac{a}{b} = \dfrac{m + \sqrt{m^2 - 1}}{m - \sqrt{m^2 - 1}}$.

12. If y be the harmonic mean between x and z, where x and z are the arithmetic and geometric means between a and b: then will $\frac{1}{2}y = (a+b) \div \left\{\left(\dfrac{a}{b}\right)^{\frac{1}{4}} + \left(\dfrac{b}{a}\right)^{\frac{1}{4}}\right\}^2$.

13. If $A : a = B : b = C : c =$ &c., prove that

$$\frac{A^{3n} - a^{3n}}{A^n - a^n} + \frac{B^{3n} - b^{3n}}{B^n - b^n} + \frac{C^{3n} - c^{3n}}{C^n - c^n} + \text{\&c.}$$

$$= \frac{(A + B + C + \text{\&c.})^{3n} - (a + b + c + \text{\&c.})^{3n}}{(A + B + C + \text{\&c.})^n - (a + b + c + \text{\&c.})^n}.$$

14. If $a + b \propto a - b$, it is required to prove that $a^2 + b^2 \propto ab$, and $a^3 + b^3 \propto ab(a \pm b)$.

15. If $y = p + q$, where $p \propto x$ and $q \propto \dfrac{1}{x}$: also, when $x = 1$, $y = 6$, and when $x = 2$, $y = 5$: prove that $y = \dfrac{4}{3}x + \dfrac{14}{3x}$.

16. Prove that the ratio $a\left(1 - \dfrac{2c}{a}\right) : \left(1 - \dfrac{2cx}{a}\right)^{\frac{3}{2}}$ is nearly $a - 2c + 3cx : 1$, if c be small with respect to a.

17. If the prices of two trees containing p and q cubic feet of timber be a£ and b£: required the price of a tree containing r cubic feet, when the values of the timber and bark are respectively proportional to the m^{th} and n^{th} powers of the quantity of timber in the tree.

Answer: $\dfrac{abr(1 + r^{m-n})(p^{m-n} - q^{m-n})}{aq(1+q^{m-n})(p^{m-n} - r^{m-n}) + bp(1+p^{m-n})(r^{m-n} - q^{m-n})}$.

ARITHMETICAL PROGRESSION.

1. Prove the truth of the following results.

(1) The 10^{th} term of $2 + 5 + 8 + \&c.$ is 29, and the sum of 10 terms is 155.

(2) The 13^{th} term of $3 + 9 + 15 + \&c.$ is 75, and the sum of 13 terms is 507.

(3) The 24^{th} term of $7 + 5 + 3 + \&c.$ is -39, and the sum of 24 terms is -384.

(4) The 20^{th} term of $4 - 3 - 10 - 17 - \&c.$ is -129, and the sum of 20 terms is -1250.

(5) The 12^{th} term of $1 + \frac{3}{2} + 2 + \frac{5}{2} + \&c.$ is $6\frac{1}{2}$, and the sum of 12 terms is 45.

(6) The 6^{th} term of $\frac{2}{3} + \frac{7}{15} + \frac{4}{15} + \&c.$ is $-\frac{1}{3}$, and the sum of 6 terms is 1.

(7) The n^{th} term of $\frac{1}{3} + \frac{5}{6} + \frac{4}{3} + \&c.$ is $\frac{1}{2}n - \frac{1}{6}$, and the sum of n terms is $\frac{1}{12}n + \frac{1}{4}n^2$.

(8) The n^{th} term of $\frac{n-1}{n} + \frac{n-2}{n} + \frac{n-3}{n} + \&c.$ is 0, and the sum of n terms is $\frac{1}{2}(n-1)$.

(9) The sum of n terms of $\frac{a-b}{a+b} + \frac{3a-2b}{a+b} + \&c.$ is

$$\left\{na - \tfrac{1}{2}(n+1)b\right\} \frac{n}{a+b}.$$

2. The first term is $n^2 - n + 1$, and the common difference is 2: prove that the sum of n terms is n^3: and thence shew that

$$1^3 = 1, \quad 2^3 = 3 + 5, \quad 3^3 = 7 + 9 + 11, \quad \&c.$$

3. The sum of the first two terms of an arithmetical progression is 4, and the fifth term is 9: find the series.

Answer: 1, 3, 5, 7, 9, &c.

4. The first two terms of an arithmetical progression being together $= 18$, and the next three terms $= 12$: how many terms must be taken to make 28?

Answer: 4 or 7.

5. The latter half of $2n$ terms of an arithmetical series is equal to one-third of the sum of $3n$ terms of the same series.

6. The difference between the sum of m and n terms of an arithmetical progression: the sum of $m+n$ terms

$$= m - n : m + n.$$

7. The sum of n terms of an increasing arithmetical progression whose common difference is equal to the least term, is the sum of $n+1$ magnitudes, each of which is half the greatest term.

8. The sum of an even number of terms of an arithmetical progression whose common difference is equal to the least term, will be four times the sum of half that number of terms diminished by half the last term.

9. There are p arithmetic progressions each beginning with 1, and the common differences are 1, 2, 3, &c., p: shew that the sum of their n^{th} terms $= \frac{1}{2}\{(n-1)p^2 + (n+1)p\}$.

10. If s_1, s_2, s_3, &c., s_p be the sums of p arithmetical progressions each continued to n terms: the first terms being 1, 2, 3, &c. and the common differences 1, 3, 5, &c.: then wil $s_1 + s_2 + s_3 + \text{\&c.} + s_p = \frac{1}{2}np(np+1)$.

11. Prove that 1, 3, 5, 7, &c., is the only arithmetic progression beginning with 1, in which the sum of the fi half of any even number of terms has to the sum of t last half the same constant ratio.

12. If the p^{th} and q^{th} terms of an arithmetical progression be P and Q: prove that the first term

$$= \frac{Q(p-1) - P(q-1)}{p-q}:$$

and that the sum of n terms

$$= \tfrac{1}{2} n \left\{ \frac{Q(2p-n-1) - P(2q-n-1)}{p-q} \right\}.$$

13. Determine the relation between a, b and c, so that they may be the p^{th}, q^{th} and r^{th} terms of an arithmetical progression.

Answer: $(q-r)a + (r-p)b + (p-q)c = 0$.

14. Insert four arithmetic means between 193 and 443: and three between 117 and 477.

Answer: 243, 293, 343, 393: and 207, 297, 387.

15. The sum of m arithmetic means between 1 and 19: the sum of the first $m-2$ of them :: 5 : 3: prove that their number is 8.

16. There are m arithmetic means between 1 and 31: and the 7^{th} : $(m-1)^{\text{th}}$ :: 5 : 9: shew that the number of means is 14.

17. In an arithmetical progression, if the $(p+q)^{\text{th}}$ term $=m$, and the $(p-q)^{\text{th}}$ term $=n$: then the p^{th} term $= \frac{1}{2}(m+n)$, and the q^{th} term $= m - \frac{1}{2}(m-n)\frac{p}{q}$.

18. If s_n, s_{n+1}, s_{n+2}, &c., denote the sums of n, $n+1$, $n+2$, &c. terms of an arithmetical progression: prove that $s_n + s_{n+1} + s_{n+2} +$ &c. to n terms

$$= n(3n-1)\frac{a}{1.2} + n(n-1)(7n-2)\frac{d}{1.2.3}.$$

19. If s_1, s_2, s_3, &c., s_{2n} be the sums of n terms of $2n$ arithmetical progressions, whose first terms are the same, and common differences d, $2d$, $3d$, &c., $2nd$: then will

$$(s_2 + s_4 + \text{\&c.} + s_{2n}) - (s_1 + s_3 + \text{\&c.} + s_{2n-1}) = \tfrac{1}{2} n^2 (n-1) d.$$

20. Prove that $a_1 a_2 + a_2 a_3 + \&c. + a_{n-1} a_n$, may always be expressed in finite terms, when a_1, a_2, a_3, &c., are in arithmetical progression.

21. If s_m denote the sum of m terms of an arithmetical progression, then will $s_{n^2} = \frac{1}{2} n(n-1) s_{2n} - n(n-2) s_n$.

22. If s_n denote the sum of n terms of an arithmetical progression, prove that

$$(q-r)\, qr\, s_{pm} + (r-p)\, pr\, s_{qm} + (p-q)\, pq\, s_{rm} = 0.$$

GEOMETRICAL PROGRESSION.

1. Prove the correctness of the following results.

(1) The n^{th} term of $1 + 3 + 9 + \&c.$ is 3^{n-1}, and the sum of n terms is $\frac{1}{2}(3^n - 1)$.

(2) The n^{th} term of $1 - 2 + 2^2 - \&c.$ is $\pm 2^{n-1}$, and the sum of n terms is $\frac{1}{3}(1 \mp 2^n)$.

(3) The n^{th} term of $\frac{1}{3} + \frac{1}{2} + \frac{3}{4} + \&c.$ is $\frac{3^{n-2}}{2^{n-1}}$, and the sum of n terms is $\frac{1}{3}\left(\frac{3^n - 2^n}{2^{n-1}}\right)$.

(4) The n^{th} term of $3\frac{3}{8} + 2\frac{1}{4} + 1\frac{1}{2} + \&c.$ is $\left(\frac{2}{3}\right)^{n-4}$, and the sum of n terms is $\frac{1}{8}\left(\frac{3^n - 2^n}{3^{n-4}}\right)$.

(5) The n^{th} term of $\frac{1}{5} - \frac{2}{15} + \frac{4}{45} - \&c.$ is $\pm \frac{2^{n-1}}{5 \cdot 3^{n-1}}$, and the sum of n terms is $\frac{1}{25}\left(\frac{3^n \mp 2^n}{3^{n-1}}\right)$.

(6) The n^{th} term of $\frac{1}{\sqrt{2}} + \frac{1}{2} + \frac{1}{2\sqrt{2}} + \&c.$ is $\frac{1}{\sqrt{2^n}}$, and the sum of n terms is $\frac{1}{\sqrt{2^n}}\left(\frac{\sqrt{2^n} - 1}{\sqrt{2} - 1}\right)$.

(7) The sum of $\frac{1}{2}+\frac{1}{3}+\frac{2}{9}+$ &c. to ∞ is $1\frac{1}{3}$.

(8) The sum of $\frac{3}{2}-1+\frac{2}{3}-$ &c. to ∞ is $\frac{9}{10}$.

(9) The sum of $2\frac{1}{2}+\frac{1}{2}+\frac{1}{10}+$ &c. to ∞ is $3\frac{1}{8}$.

(10) The sum of $\sqrt{\frac{3}{2}}+\sqrt{\frac{2}{3}}+\frac{2}{3}\sqrt{\frac{2}{3}}+$ &c. to n terms is $\sqrt{\frac{3}{2}}\left(\frac{3^n-2^n}{3^{n-1}}\right)$, and to ∞ is $3\sqrt{\frac{3}{2}}$.

(11) The sum of $1-2x+2x^2-2x^3+$ &c. to ∞ is $\frac{1-x}{1+x}$.

(12) The sum of $\frac{1}{3}+\frac{1}{6\sqrt{-1}}-\frac{1}{12}-$&c. to ∞ is $\frac{2\sqrt{-1}}{6\sqrt{-1}-3}$.

2. In any geometrical progression, consisting of an odd number of terms, the sum of the squares of the terms is equal to the sum of all the terms multiplied by the excess of the odd terms above the even.

3. In any geometrical progression, the sum of the first and last terms is greater than the sum of any other two terms equidistant from the extremes.

4. In any geometrical progression of an even number of terms, the sum of the odd terms is to the sum of the even terms as 1 to r.

5. If s_1, s_2, s_3, &c., s_n, be the sums of n geometrical progressions, whose first terms are a, $2a$, $3a$, &c., na: then will $s_1+s_2+s_3+\text{\&c.}+s_n=\frac{n(n+1)}{2}\left(\frac{r^n-1}{r-1}\right)a$.

6. If a, b, c, d, &c., be n quantities in geometrical progression: then will $\frac{1}{a^2-b^2}$, $\frac{1}{b^2-c^2}$, $\frac{1}{c^2-d^2}$, &c., be in geometrical progression: and the sum of n terms will be

$$\frac{1}{b^{2(n-1)}}\,\frac{a^{2n}-b^{2n}}{(a^2-b^2)^2}.$$

7. In a geometrical progression, if the $(p+q)^{\text{th}}$ term $= m$, and the $(p-q)^{\text{th}} = n$: then will the p^{th} term $= \sqrt{mn}$, and the q^{th} term $= m\left(\frac{n}{m}\right)^{\frac{p}{2q}}$.

8. If the p^{th} and q^{th} terms of a geometrical progression be P and Q, then will the n^{th} term be $\left(\frac{Q^{p-n}}{P^{q-n}}\right)^{\frac{1}{p-q}}$, and the sum of n terms

$$= \left(\frac{Q^{p-n}}{P^{q-1}}\right)^{\frac{1}{p-q}}\left\{\frac{P^{\frac{n}{p-q}} - Q^{\frac{n}{p-q}}}{P^{\frac{1}{p-q}} - Q^{\frac{1}{p-q}}}\right\}.$$

9. If there be n quantities in geometrical progression, whose common ratio is r, and s_m denote the sum of the first m terms: prove that the sum of their products, taken two and two together, $= \frac{r}{r+1}s_n s_{n-1}$.

10. If s be the sum of n terms of a geometrical progression, whose first term is t_1, and t_2, t_3, &c., t_n, denote the sums of the first two, three, four, &c., terms, then will

$$(s+t_1) + (s+t_2) + \text{\&c.} = \frac{t_1}{r-1}\left\{n(r^n - 2) + \frac{r}{r-1}(r^n - 1)\right\}.$$

11. Insert three geometric means between 39 and 3159: also, between 37 and 2997.

Answer: 117, 351, 1053: and 111, 333, 999.

12. If $s = (x-y) + \left(\frac{y^2}{x} - \frac{y^3}{x^2}\right) + \text{\&c.}$ to n terms, and σ denote the sum *in infinitum:* then will

$$s : \sigma = x^{2n} - y^{2n} : x^{2n}.$$

13. If σ_1 represent the sum of an infinite geometrical progression, σ_2 the sum of the squares, σ_3 the sum of the cubes, &c., of the terms: then will

$$\frac{1}{\sigma_1} \pm \frac{1}{\sigma_2} + \frac{1}{\sigma_3} \pm \text{\&c. } \textit{in infinitum} = \frac{1}{a \mp 1} - \frac{r}{a \mp r}.$$

14. If σ_1, σ_2, σ_3, &c., σ_n be the sums of n infinite geometrical progressions, the first term of each being 1, and the common ratios $\frac{1}{r}$, $\frac{1}{r^2}$, $\frac{1}{r^3}$, &c., $\frac{1}{r^n}$: then will

$$\frac{1}{\sigma_1} + \frac{1}{\sigma_2} + \frac{1}{\sigma_3} + \text{\&c.} + \frac{1}{\sigma_n} = n - \frac{r^n - 1}{r^n (r-1)}.$$

15. If σ_1, σ_2, σ_3, &c., denote the sums of an infinite number of infinite geometrical progressions, whose first terms are a, a^2, a^3, &c., and common ratios r, $2r$, $3r$, &c., then will

$$\frac{1}{\sigma_1} + \frac{1}{\sigma_2} + \frac{1}{\sigma_3} + \text{\&c.} = \frac{a(1-r) - 1}{(a-1)^2}.$$

16. Between $n+1$ quantities, (x, y), $(x, 2y)$, $(x, 4y)$, &c., are inserted n geometric means, and M_1, M_2, M_3, &c., are the n^{th} means respectively: then will

$$\frac{M_1}{M_2} + \frac{M_2}{M_3} + \frac{M_3}{M_4} + \text{\&c.} = \left(\frac{n^{n+1}}{2^n}\right)^{\frac{1}{n+1}}.$$

17. If a, b, c, d be in geometrical progression, prove that $(a + 3b + 3c + d)\,bc = (b + c)^3$, and

$$(a + b + c + d)^2 = (a + b)^2 + (c + d)^2 + 2(b + c)^2.$$

18. The n^{th} term of $2 + 6 + 14 + 30 + \text{\&c.}$ is $2^{n+1} - 2$, and the sum of n terms is $2^{n+2} - (2n + 4)$.

19. The n^{th} term of $4 + 10 + 28 + 82 + \text{\&c.}$ is $3^n + 1$, and the sum of n terms is $\frac{1}{2} 3^{n+1} + n - \frac{3}{2}$.

20. The n^{th} term of $3 + 6 + 11 + 20 + \text{\&c.}$ is $2^n + n$, and the sum of n terms is $2^{n+1} + \frac{1}{2}(n^2 + n - 4)$.

21. If σ be the sum, and σ_2 the sum of the squares of the terms of an infinite geometrical progression: then will

$$a = \frac{2\sigma\sigma_2}{\sigma^2 + \sigma_2}, \quad \text{and} \quad r = \frac{\sigma^2 - \sigma_2}{\sigma^2 + \sigma_2}.$$

HARMONICAL PROGRESSION.

1. Insert four harmonic means between 2 and 12.

Answer: $2\frac{2}{5}$, 3, 4, and 6.

2. Insert six harmonic means between 3 and $\frac{6}{23}$.

Answer: $\frac{6}{5}, \frac{3}{4}, \frac{6}{11}, \frac{3}{7}, \frac{6}{17}, \frac{3}{10}$.

3. Shew that 24 is a fourth harmonical proportional to 6, 8, and 12.

4. The sum of three terms of a harmonic series is $1\frac{1}{12}$, and the first term is $\frac{1}{2}$: find the series.

Answer: $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, &c.

5. The sum of three numbers in harmonical progression is 11, and the sum of their squares is 49: find them.

Answer: 2, 3, 6.

6. There are four numbers, the first three of which are in arithmetical progression, and the last three in harmonical: prove that the products of the extremes and means are equal.

7. If $a^x = b^y = c^z =$ &c., and a, b, c, &c., be in geometrical progression, then will x, y, z, &c., be in harmonical progression.

INDETERMINATE COEFFICIENTS.

1. By the method of indeterminate coefficients prove,

$$\frac{1+x}{1-x-x^2} = 1 + 2x + 3x^2 + 5x^3 + 8x^4 + \text{\&c.}$$

2. $$\frac{1+x}{(1-x)^2} = 1 + 3x + 5x^2 + 7x^3 + 9x^4 + \text{\&c.}$$

3. $$\frac{1+x}{(1-x)^3} = 1^2 + 2^2x + 3^2x^2 + 4^2x^3 + 5^2x^4 + \text{\&c.}$$

4. $$\frac{1-3x+2x^2}{1+x+x^2} = 1 - 4x + 5x^2 - x^3 - \text{\&c.}$$

5. $$\frac{1-x}{1-2x-3x^2} = 1 + x + 5x^2 + 13x^3 + \text{\&c.}$$

7. $\left(\frac{1+2x}{1+x}\right)^m = 1 + m\left(\frac{x}{1+2x}\right) + \frac{m(m+1)}{1 \cdot 2}\left(\frac{x}{1+2x}\right)^2 + \&c.$

8. The cube roots of $7 + 5\sqrt{2}$ and $11\sqrt{2} + 3\sqrt{27}$, are $\sqrt{2} + 1$ and $\sqrt{2} + \sqrt{3}$ respectively.

9. If a_1, a_2, a_3, a_4 be any successive coefficients of the expanded binomial, then will

$$(a_2 a_3 + a_1 a_4)(a_2 - a_3) = 2a_1 a_3^2 - 2a_4 a_2^2.$$

10. If N be the n^{th} term of the expansion of $(1 + x)^m$, then will the series after the first n terms be

$$Nx\left(1 - \frac{m+1}{n}\right) + Nx^2\left(1 - \frac{m+1}{n}\right)\left(1 - \frac{m+1}{n+1}\right)$$

$$+ Nx^3\left(1 - \frac{m+1}{n}\right)\left(1 - \frac{m+1}{n+1}\right)\left(1 - \frac{m+1}{n+2}\right) + \&c.$$

VARIATIONS AND COMBINATIONS.

1. The number of variations two together : the number three together = 1 : 5 : find the number of things.

Answer : 7.

2. The number of things : the number of variations three together = 1 : 20 : what is that number?

Answer : 6.

3. The number of combinations of m things taken four together : the number taken two together = 15 : 2 : find the value of m.

Answer : 12.

4. There is a certain number of things of which the variations taken eight together = 80, and taken ten together = 960 : how many must be taken away from the original number, that of the remaining things taken two together, the [com]binations may be 15.

Answer : 6.

6. $\dfrac{x^2}{x^2 + 2ax + a^2} = 1 - \dfrac{2a}{x} + \dfrac{3a^2}{x^2} - \dfrac{4a^3}{x^3} + \text{\&c.}$

7. $\dfrac{6x^2 - 4x - 6}{(x-1)(x-2)(x-3)} = \dfrac{1}{x-1} + \dfrac{2}{x-2} + \dfrac{3}{x-3}.$

8. $\dfrac{x^2}{(x+1)(x+2)(x+3)} = \dfrac{1}{2(x+1)} - \dfrac{4}{x+2} + \dfrac{9}{2(x+3)}.$

9. $\dfrac{x+2}{x^3 - x} = \dfrac{1}{2(x+1)} + \dfrac{3}{2(x-1)} - \dfrac{2}{x}.$

10. $\dfrac{13 + 21x + 2x^2}{1 - 5x^2 + 4x^4} = \dfrac{1}{1+x} - \dfrac{6}{1-x} + \dfrac{2}{1+2x} + \dfrac{16}{1-2x}.$

THE BINOMIAL THEOREM.

1. If S be the sum of the odd terms of the expansion of $(a+x)^m$, and s the sum of the even terms: then will

$$S^2 - s^2 = (a^2 - x^2)^m.$$

2. Prove that four times the product of the sums of the odd and even terms of the expansion of $(a+x)^m$, is equal to

$$(a+x)^{2m} - (a-x)^{2m}.$$

3. If the coefficients of the expansion of $(a-x)^m$, be multiplied by a, $a+b$, $a+2b$, &c. in order: it is required to prove that the result $= 0$.

4. Prove that if the coefficients of the expansion of $(a-x)^m$ be multiplied by 1^n, 2^n, 3^n, &c., in order, the result $= 0$, if m be greater than n.

5. If S be the sum of all the coefficients of the expansion of $(a+x)^{2m}$, and C the coefficient of the middle term: it is required to prove that

$$\{1.3.5.\text{\&c.}(2m-1)\}\,S = \{2.4.6.\text{\&c.}\,2m\}\,C.$$

6. $\left(\dfrac{1+x}{1-x}\right)^m = 1 + m\left(\dfrac{2x}{1+x}\right) + \dfrac{m(m+1)}{1.2}\left(\dfrac{2x}{1+x}\right)^2 + \text{\&c.}$

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Answer: 6.

7. In any system of notation, whose local value is r, in any multiple of $r-1$, the sum of the digits is either equal to $r-1$, or to some multiple of it.

8. If any number, a multiple of 11, and a number consisting of the same digits in an inverted order, be each divided by 11, the sums of the digits in the two quotients are equal.

9. If the sum of the odd digits in a number be $11m+c$, and of the even $11n+c$, this number being divided successively by 11 and 9, leaves the same remainder as $m+n+c$ when divided by 9.

10. In resolving an irreducible fraction $\frac{a}{b}$ into a recurring decimal, prove that when any two remainders give the divisor b for their sum, the two consecutive remainders give the same sum: and the sum of the two figures in the period, which correspond to these remainders, is 9.

11. If $\frac{a}{b}$ be an irreducible fraction, and b be any number except 9 or 3, shew that when $\frac{a}{b}$ is converted into a recurring decimal, the period will be divisible by 9, and the sum of the remainders a multiple of b.

12. Prove that the sum of all the numbers of n places which can be formed with the n digits a, b, c, &c.: the sum of all the numbers of n places which can be formed with the n digits p, q, r, &c. of the same scale

$$= a + b + c + \&c. : p + q + r + \&c.$$

13. There is a number consisting of three digits in geometrical progression : the number : the sum of its digits $= 124 : 7$, and if 594 be added to it, the digits will be inverted: what is it?

Answer: 248.

14. A certain number when expressed in the scale of r consists of two digits: and when divided in that scale by the digit on the right, gives a quotient 27 and a remainder 2: but when divided by 9, gives a quotient equal to three times the figure on the right and a remainder 2: and lastly, when expressed in the scale of 25, the digits will be inverted: what is the number?

Answer: 83.

15. There is a certain number, which, when expressed in the scale of r, is 39: but when expressed in the scale of $r-1$, consists of two digits, so that when divided by the difference of the digits, the quotient is 21: but, when divided by the sum of the digits, the quotient increased by 17, will give the number with its digits inverted: find the number.

Answer: 42.

PROPERTIES OF NUMBERS.

1. If n be a whole number, prove that n^3+5n is divisible by 6.

2. If n be any whole number, then will $n(n^2-1)(n^2-4)$ be divisible by 120.

3. If n be any odd number, then n^5-n will be a multiple of 12.

4. If n be an even number, then will $n(n^2+20)$ be divisible by 48.

5. The square of every odd number diminished by 1 is divisible by 8.

6. If from the cube of any even number be subtracted four times the number itself, the remainder will be a multiple of 48.

7. If n be an even number, then will $n^2(n^2-4)$ be divisible by 192: and if m be odd, $m^2(m^2-4)(m^2-9)$ will be divisible by 1920.

8. If an odd and even square number be added together, and the sum be also a square number: then the even square is a multiple of 16.

9. If any square number be divided by 12, the remainder is also a square number.

10. Supposing the sum of 51 cards in a common pack to be $10n + a$, a being less than 10: prove that the value of the last card is $10 - a$, the court cards reckoning for 10, and the aces, deuces, &c. for 1, 2, &c.: and find the value of n.

Answer: 33.

11. Prove that $(n+1)(n+2)(n+3)$ &c. to n factors

$$= 2^n (1 \times 3 \times 5 \times \text{\&c. to } n \text{ factors}).$$

12. Find the number of divisors of 2160, and also their sum.

Answer: 40 and 7440.

13. What number multiplied by 48, will make it a complete fourth power?

Answer: 27.

14. The square of every prime number greater than 3, diminished by 1, is divisible by 24.

15. The sum of any number of prime numbers, in arithmetical progression, is a composite number.

16. If a and b be prime numbers, the number of numbers prime to ab and less than ab is $(a-1)(b-1)$, unity being considered one of them.

17. If we divide a, a^2, a^3, &c. by a prime number p, we shall obtain a remainder $= 1$, before we have taken p terms: also, after this remainder, the remainders recur.

18. If a be a prime number, and b any other number prime to a, shew that if b^2, $(2b)^2$, $(3b)^2$, &c., $\{\frac{1}{2}(a-1)b\}^2$ be divided by a, they will each leave a different remainder.

19. Prove that the sum of any two consecutive triangular numbers is a square number.

20. Shew that the ratio between a triangular and square number of the same root, approaches to $\frac{1}{2}$ as that root is increased.

21. Prove that the sum of n terms of the series

$$1^3 + 3^3 + 5^3 + 7^3 + \&c.$$

is a hexagonal number, whose root is n^2.

22. If n be a pentagonal number, prove that $24n + 1$ is a square number.

23. If n be a heptagonal number, then will $40n + 9$ be a square number.

24. If n be an r-gonal number, it is required to shew that $8(r - 2)n + (r - 4)^2$ is a square number.

THE END.